Near critical dimers and massive SLE

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(joint work with Levi Haunschmid-Sibitz)

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Outline

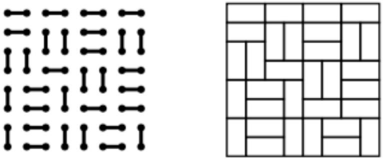
1) Reminders about the classical *dimer model*
2) *Near-critical* weights
   - Definition, non Gaussian scaling limit
   - Equivalent representation: *Temperley's bijection*
   - New results: convergence to *massive SLE*, universality, conformal covariance,
   - Along the way: scaling limit of LERW with drift
3) An exact discrete Girsanov theorem on the triangular lattice
4) Some open questions: Sine-Gordon, Ising etc?

1) The dimer model

The dimer model

Let G be a finite, planar, bipartite graph.

A *dimer cover* (or *perfect matching*): a set of edges (=dimers), such that each vertex is incident to exactly one dimer.



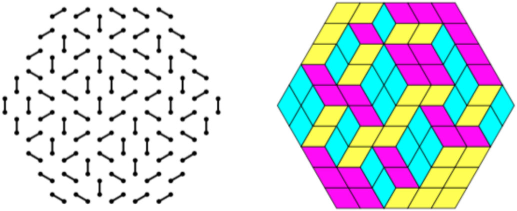
The *dimer model* with edge weights w_e:

$$\mathbb{P}(\mathbf{m}) = \frac{1}{Z} \prod_{e \in \mathbf{m}} w_e.$$

Typically $w_e \equiv 1$ ($\to$ *critical*!)

The dimer model as a random surface

Honeycomb lattice: *lozenge tiling* or a stack of cubes



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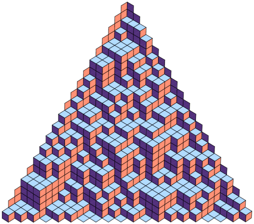
Height function

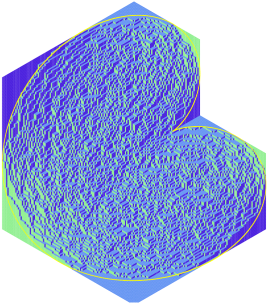
Introduced by Thurston. Hence view as a random surface.

Note: depends on the choice of a reference frame.

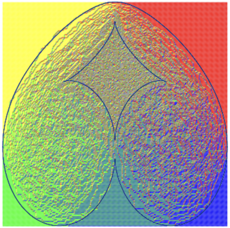
Large scale behaviour?



Kenyon



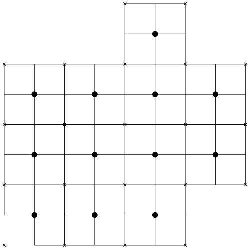
Kenyon–Okounkov–
Sheffield 2006



The effect of boundary conditions is, however, not entirely trivial and will be discussed in more detail in a subsequent paper.

P. W. Kasteleyn, 1961

Temperleyan boundary conditions



Divide the vertices into black and white.
Divide further into $B_0 = \bullet, B_1 = \times$
(and W_0, W_1).

Temperleyan: all corners are $B_1 = \times$, and one corner is removed.

Scaling limit of height function

Theorem (Kenyon '99)

Let $\mathcal{D} \subset \mathbb{C}$ bounded domain, $\mathcal{D}^\delta = \mathcal{D} \cap \delta\mathbb{Z}^2$ with *Temperleyan* boundary conditions. Let h^δ be the associated height function. Then,

$$h^\delta - \mathbb{E}(h^\delta) \to \frac{1}{\sqrt{\pi}} h_{\mathcal{D}}^{\mathrm{GFF}} \qquad \text{as} \qquad \delta \to 0,$$

in distribution.

Main ingredients of the proof:

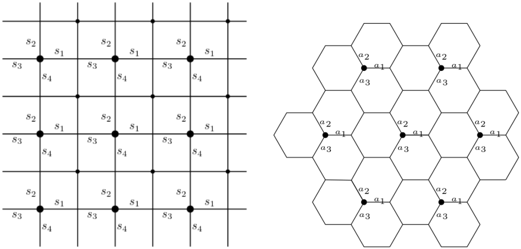
- Kasteleyn theory (exact solvability): dimer correlations are given by determinants of inverse *Kasteleyn matrix*,
- Asymptotic computation of inverse Kasteleyn matrix (discrete holomorphic + boundary conditions)
- Computation of moments

2) Near-critical dimer model

Makarov–Smirnov (2009):

The key property of SLE is its conformal invariance, which is expected in 2D lattice models only at criticality, and the question naturally arises: Can SLE success be replicated for off-critical models? In most off-critical cases to obtain a non-trivial scaling limit one has to adjust some parameter [...], sending it at an appropriate speed to the critical value. Such limits lead to massive field theories...,

Biperiodic setup

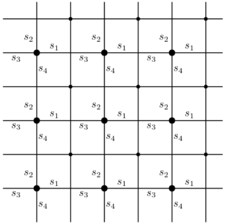


Choose $s_i = 1 + c_i\delta$, where δ = mesh size. *Gasesous/Liquid boundary...*

Massive Laplacian

Let K = Kasteleyn matrix, $D = KK^*$. Then D is (essentially) a *massive Laplacian*:

$$D(b,b) = -\sum_{i=1}^{4} s_i^2$$



but

$$\sum_{b'} D(b,b') = 2s_2s_4 + 2s_1s_3 < |D(b,b)|$$

by AM-GM.

Describes a *massive walk* (fixed killing probability).

Natural guess:

Scaling limit = Massive GFF?

$$\mathbb{E}[h(x)h(y)] = \int_0^\infty e^{-m^2 t} p_t(x,y)\,dt$$

Negative answer

Unfortunately this guess is wrong.

Theorem (Chhita, 2012)

Limiting moments of height function can be computed; no Wick rule so non Gaussian !

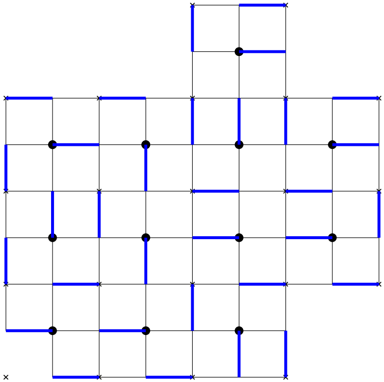
New results for near-critical dimers

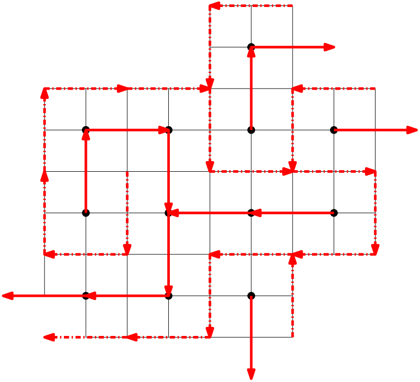
With Levi Haunschmid (2022) we prove:

- Exact connection with Makarov and Smirnov's *massive* SLE_2 (and with massive Laplacian).
- Existence and universality of *scaling limit* of height function in Temperleyan domains
- Conformal *covariance* of scaling limit

Temperley's bijection

Dimers on $\mathbb{Z}^2 \cap D \leftrightarrow$ pairs of dual spanning trees on B_0, B_1 lattices.
Temperleyan boundary conditions: wired/free tree.



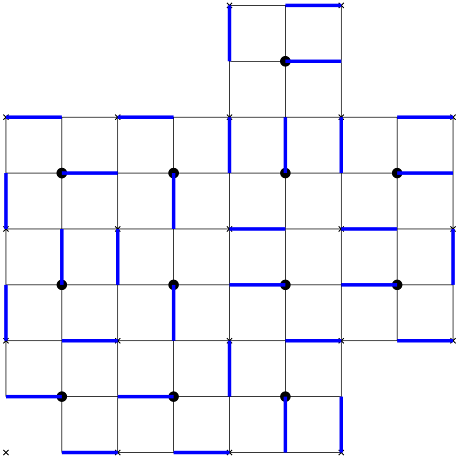


B.–Laslier–Ray point of view

Often easier to work with Temperleyan trees.
Keeps all the information, even in scaling limit ("Imaginary Geometry").

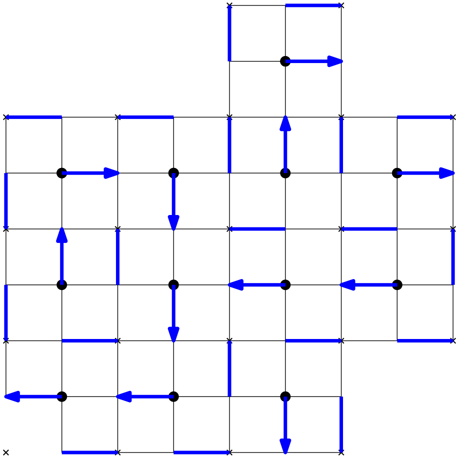
Temperley's bijection 1

Dimers on $\mathbb{Z}^2 \cap D$, Temperleyan boundary conditions.



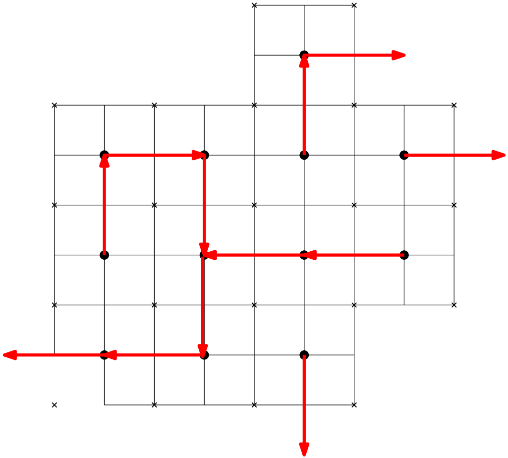
Temperley's bijection 2

Orient dimers black $\rightarrow$ white (just $B_0 = \bullet$ for now)



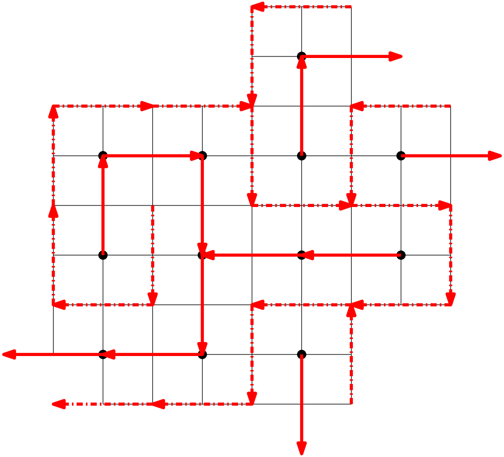
Temperley's bijection 3

Double each oriented dimer to get spanning tree on B_0 lattice (wired boundary conditions).



Temperley's bijection 4

On B_1 lattice, get dual (free boundary conditions) spanning tree.



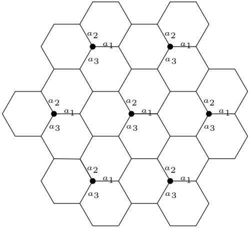
Remarks

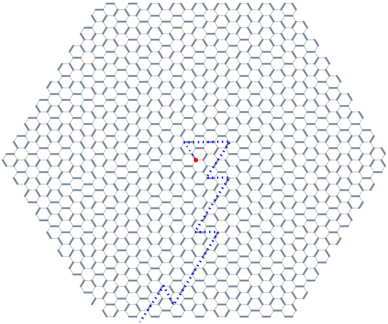
- The bijection is local.
- Temperleyan boundary conditions for dimer $\Rightarrow$ wired/free boundary conditions for trees.
- If $w_e \equiv 1$ then $(\mathcal{T}, \mathcal{T}^\dagger)$ uniform.
- More generally, in biperiodic setup,

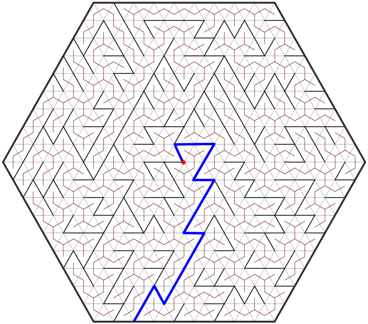
$$\mathbb{P}(\mathcal{T} = \mathbf{t}) \propto \prod_{e \in \mathbf{t}} w_{e/2}.$$

Owing to biperiodic structure, (directed) edges of $\mathcal{T}$ come with weight $s_1, \ldots, s_4$.

Extends to hexagonal lattice







Scaling limit of Temperleyan tree

Consider off-critical dimer model on square with $s_i = 1 + c_i\delta$.
Let $\mathcal{T}$ = Temperleyan B_0-tree.

$$\mathbb{P}(\mathcal{T} = \mathbf{t}) \propto \prod_{v \in B_0} s_v(\mathbf{t})$$

where $s_v(\mathbf{t}) \in \{s_1, \ldots, s_4\}$ depending on the direction of the unique outgoing edge from v in $\mathbf{t}$.

Wilson's algorithm

The branch connecting z to ∂D is LERW for the random walk on B_0 with jump probabilities $(s_i)_{i=1}^4$.

The random walk itself converges to BM with drift α,

$$\alpha = \frac{1}{4}(c_1 + c_2 i + c_3 i^2 + c_4 i^3)$$

But what is the scaling limit of LERW?

Connection with massive SLE_2

Suppose

$$c_1 + c_3 = c_2 + c_4 = 0$$

Theorem 1 (B.–Haunschmid)

Let $z \in \Omega$. Let γ^δ = path in Temperleyan tree to $\partial\Omega$, Y_δ = endpoint. Then conditionally on $Y_\delta = y_\delta$,

$$\gamma^\delta \to \mathrm{mSLE}_2,$$

where mass $m = \|\alpha\|$.

Massive SLE_2

Consider random walk killed with probability $m^2\delta^2$ at each step.

Condition to leave Ω without dying. What is scaling limit of LERW?

Theorem (Makarov–Smirnov (2009), Chelkak-Wan (2019))

massive LERW converges to "massive SLE_2"

Described by Loewner's equation with driving function:

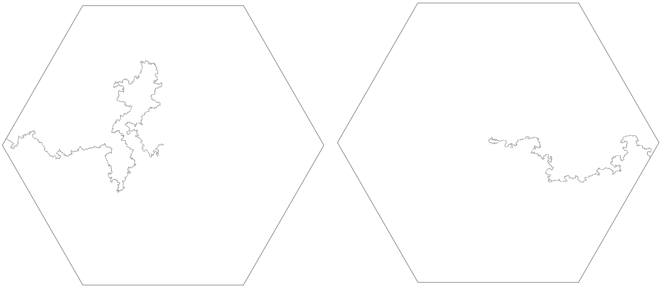
$$d\xi_t = \sqrt{2}dB_t + 2\lambda_t dt;$$

with

$$\lambda_t = \frac{\partial}{\partial w} \log \frac{P^{(m)}_{\Omega_t}(z,w)}{P^{(0)}_{\Omega_t}(z,w)} \Bigg|_{w=\gamma(t)}$$

[$m = 0$: *Lawler–Schramm–Werner 2002*]

Additional remarks



- Unconditional convergence also holds, then global Radon–Nikodym derivative:
$$\frac{d\mathbb{P}}{d\,\mathrm{mSLE}_2}(\gamma) = \exp(2\langle Y - z, \Delta\rangle)$$
where $Y =$ exit point.
- Exact same statement for hexagonal lattice $a_i = 1 + c_i\delta$,
$$\alpha = \frac{1}{3}(c_1 + c_2\tau + c_3\tau^2).$$

Convergence of height function

Corollary (B.–Haunschmid)

The Temperleyan tree $\mathcal{T}_\delta$ has a scaling limit (in Schramm topology); the limit law depends only on Δ and so is the same for hexagonal and square lattice cases.

Proof: Wilson's algorithm.

Corollary (B.–Haunschmid)

The height function of near-critical dimers in Temperleyan domains converge to the same scaling limit.

Proof: "*imaginary geometry approach*" by B.–Laslier–Ray (2020, 2019+).

Conformal covariance

Conformal covariance:

Image under conformal map preserved, up to power β of derivative of conformal map.

($\beta = 0$ means conformal invariance.)

This requires allowing for general **vector field** $\alpha : \Omega \to \mathbb{R}^2 \equiv \mathbb{C}$.

Generalised near-critical dimers

At each point $z \in B_0$, assign weights $s_i = 1 + c_i\delta$, with $c_1 + c_3 = 0, c_2 + c_4 = 0$,

$$\frac{1}{4}(c_1 + c_2 i + c_3 i^2 + c_4 i^3) = \alpha$$

Any drift vector α is uniquely encoded in this way.

The random walk with these weights converge in the scaling limit not to a Brownian motion, but to the solution of the SDE

$$dX_t = dB_t + \alpha(X_t)dt.$$

LERW with variable drift

Consider a smooth vector field $\alpha : \bar{\Omega} \to \mathbb{R}^2$. Does the LERW have a scaling limit?

Assumptions

Suppose we have sequence of planar graphs G^δ, and:

- $\alpha = \nabla \phi$ of gradient type.
- Two laws $\mathbb{P}_\delta$, $\mathbb{Q}_\delta$ such that:
  – under $\mathbb{P}_\delta$, RW converge to BM;
  – under $\mathbb{Q}_\delta$, RW converges to SDE.
- Uniform absolute continuity: i.e., $d\mathbb{Q}_\delta / d\mathbb{P}_\delta$ is Uniformly Integrable.

Holds on square lattice and triangular lattices, and conformal deformations thereof.

LERW with variable drift

Theorem 3. (B.–Haunschmid)

The loop-erased random walk with local drift α has a scaling limit.

Described by Loewner's equation with driving function in $\mathbb{D}$:

$$d\xi_t = \sqrt{2}dB_t + 2\lambda_t dt;$$

with

$$\lambda_t = \frac{\partial}{\partial w} \log \frac{P^{(\alpha)}_{\Omega_t}(z,w)}{P^{(0)}_{\Omega_t}(z,w)}\bigg|_{w=\gamma(t)}$$

where $\Omega_t = \mathbb{D} \setminus \gamma[0,t]$,
$P^{(\alpha)}_{\Omega}(z,w) =$ Poisson kernel in Ω for the SDE.

The existence of this Poisson kernel is not trivial. (Smooth Ω: Ben Arous, Kusuoka and Stroock 1984).

cf. Chelkak–Wan 2019.

Corollary

Let $\alpha = \nabla\phi$ be a smooth vector field in $\bar{\Omega}$.
Associate near-critical weights on square/hexagonal lattices.
Then height function has a scaling limit, call it $h^{(\alpha);\Omega}$. Depends just on α.

Remark: not sure what is analogue on more general lattices.

Theorem 4. (B.–Haunschmid)

Let $F : \tilde{\Omega} \to \Omega$ be a conformal map (with bounded derivative). In law,

$$h^{(\alpha);\Omega} \circ \phi = h^{(\tilde{\alpha});\tilde{\Omega}}$$

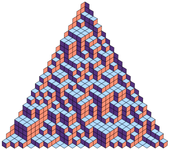
where at a point $w \in \tilde{\Omega}$,

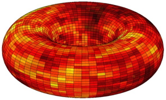
$$\tilde{\alpha}(w) = \overline{F'(w)} \cdot \alpha(F(w)).$$

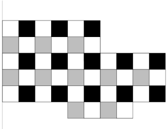
On the imaginary geometry approach to dimers

A powerful approach to dimer models:

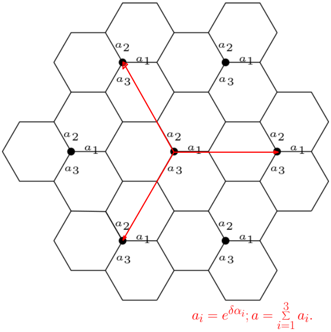
- Temperleyan & more, even for **balanced random environments** (B.–Laslier–Ray, 2020)
- Riemann Surfaces (B.–Laslier–Ray, 2019, 2022)
- Piecewise Temperleyan → **multiple SLE**$_8$ (B.–Liu, 2023)







Discrete Girsanov on triangular lattice $\mathbb{T}$.



Directed triangular lattice $\mathbb{T}$

if $\tau = e^{2i\pi/3}$,

$$\mathbb{Q}(x, x+\tau^{k-1}) = \frac{e^{\alpha_k}}{a}.$$

Define $\beta(v) > 0$ by

$$\exp(-\beta(v)^2) = (a/3)^{-3} \prod_{k=1}^{3} e^{\alpha_k},$$

well defined by AM-GM.

Discrete Girsanov on triangular lattice $\mathbb{T}$.

Define a vector $\alpha(v)$ at every vertex v in the graph,

$$\alpha = \alpha_1 + \alpha_2\tau + \alpha_3\tau^2,$$

Lemma

Fix any lattice path $\gamma = (x_0, \ldots, x_n)$ on $\mathbb{T}$.

$$\frac{\mathbb{Q}}{\mathbb{P}}(\gamma) = \exp(M_n - \tfrac{1}{2}V_n)$$

where $M_n = \frac{2}{3}\sum_{s=0}^{n-1}\langle\alpha(x_s), \mathrm{d}x_s\rangle$; and $V_n = \frac{2}{3}\sum_{s=0}^{n-1}\beta(x_s)^2$.

Discrete analogue of

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^t \Delta(X_s)\cdot \mathrm{d}X_s - \frac{1}{2}\int_0^t \|\Delta(X_s)\|^2 \mathrm{d}s\right).$$

Corollary (constant drift case)

$\mathbb{Q}_x(\cdot|x_n = y)$ is the same as a massive walk conditioned to survive up to time n and $X_n = y$.

Proof.

At each v, write $n_i = n_i(v) =$ number of times path goes in direction $1, \tau, \tau^2$.

$$
\begin{aligned}
\mathbb{Q}_x(\gamma) &= \prod_v \prod_{i=1}^{3} (\frac{e^{\alpha_i}}{a})^{n_i} \\
&= 3^{-n} \prod_v \left[((a/3)^{-3} \prod_{i=1}^{3} (e^{\alpha_i})^{\frac{n_1+n_2+n_3}{3}} \prod_{i=1}^{3} (e^{\alpha_i})^{n_i - \frac{n_1+n_2+n_3}{3}} \right] \\
&= 3^{-n} \prod_v e^{-\beta(v)^2 \frac{n_1+n_2+n_3}{3}} \exp\left(\sum_{i=1}^{3} \alpha_i (n_i - \tfrac{n_1+n_2+n_3}{3}) \right) \\
&= 3^{-n} e^{-\frac{1}{2} V_n} \exp\left(\sum_v \alpha_1 (\tfrac{2n_1-n_2-n_3}{3} + \alpha_2 (\tfrac{2n_2-n_1-n_3}{3}) + \alpha_3 (\tfrac{2n_3-n_1-n_2}{3}) \right) \\
&= 3^{-n} e^{-\frac{1}{2} V_n} \exp\left(\tfrac{2}{3} \sum_v \langle \alpha_1 + \alpha_2 \tau + \alpha_3 \tau^2, n_1 + n_2 \tau + n_3 \tau^2 \rangle \right) \\
&= 3^{-n} e^{-\frac{1}{2} V_n} \exp\left(\tfrac{2}{3} \sum_{s=0}^{n-1} \langle \alpha(x_s), \mathrm{d}x_s \rangle \right).
\end{aligned}
$$

Sine–Gordon

An integrable but non conformal QFT:

$$\mathbb{P}^{\mathrm{SG}}(dh) \propto \exp\left(z\int_D \cos(\sqrt{\beta}h(x))dx\right)\mathbb{P}^{\mathrm{GFF}}(dh),$$

where $\mathbb{P}^{\mathrm{GFF}} =$ law of $(h/\sqrt{2\pi})$, h a GFF with log correlations.

Free fermion point

$\beta = 4\pi$: **Coleman correspondence**, cf. Bauerschmidt–Webb (2023).
This is a massive extension of the fermion-boson correspondence.

At the free fermion point, the Sine-Gordon field is “particularly integrable”.

Conjectures

For $\alpha =$ constant, we conjecture convergence of the massive dimer height function to the Sine-Gordon model at the free fermion point $\beta = 4\pi$.

Progress by S. Mason (2022) (full plane in a particular case).

More generally:

Conjecture

Let $h^{(\alpha);\Omega}$ denote the limiting height function of the near-critical dimer height functions associated to the drift vector field $\alpha : \Omega \to \mathbb{R}^2$. Then the law of this field is given by

$$\mathbb{P}^{(\alpha);\Omega}(dh) \propto \exp\left(z \int_D \left\langle \alpha(x); e^{i\sqrt{\beta}h(x)} \right\rangle dx \right) \mathbb{P}^{\mathrm{GFF}}(dh),$$

as $\beta = 4\pi$. (Free fermion?)

If true, massive SLE would be “flow lines” of free fermion Sine-Gordon...