The two-periodic Aztec diamond

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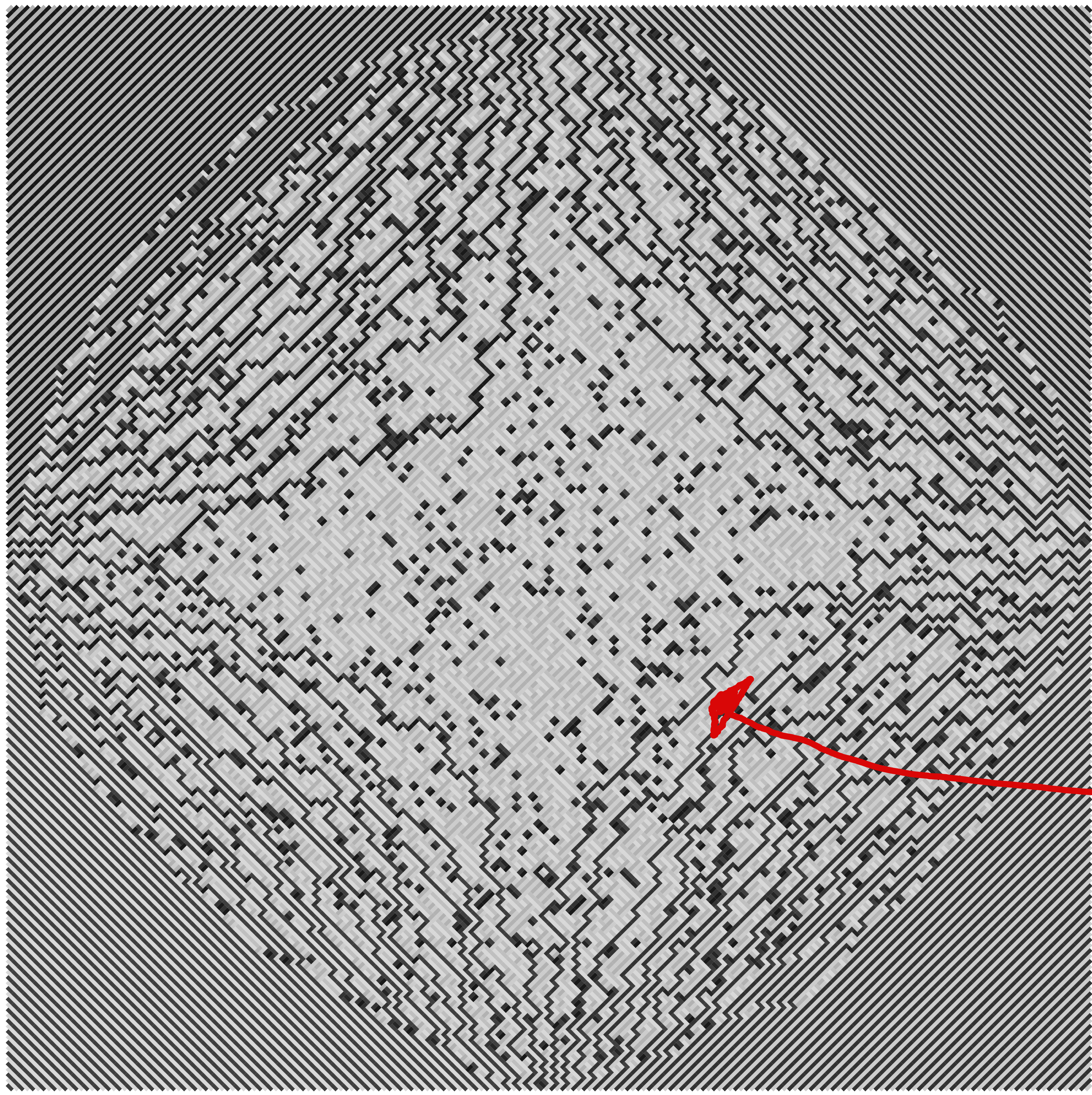
Outline



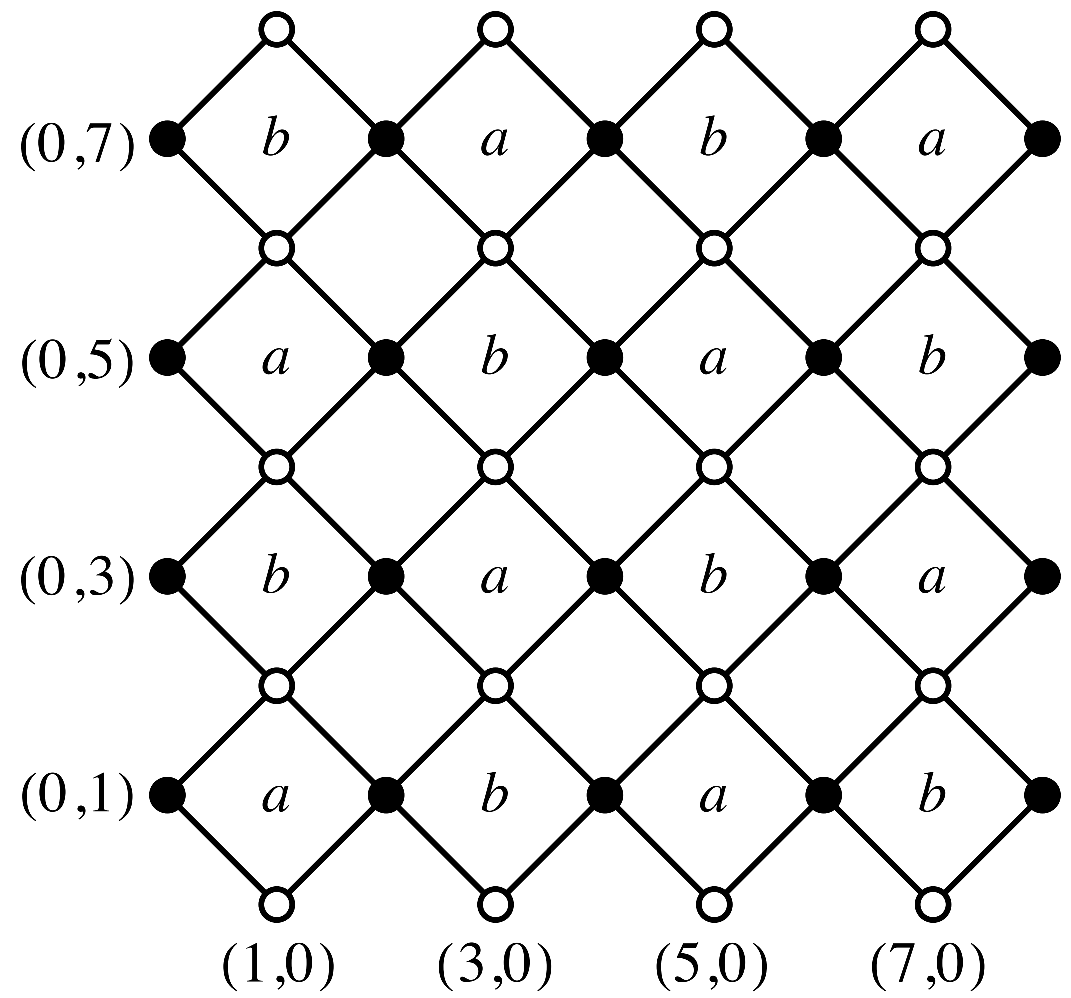
<u>Goal</u>: To describe the picture!

A realization of the two-periodic Aztec diamond.

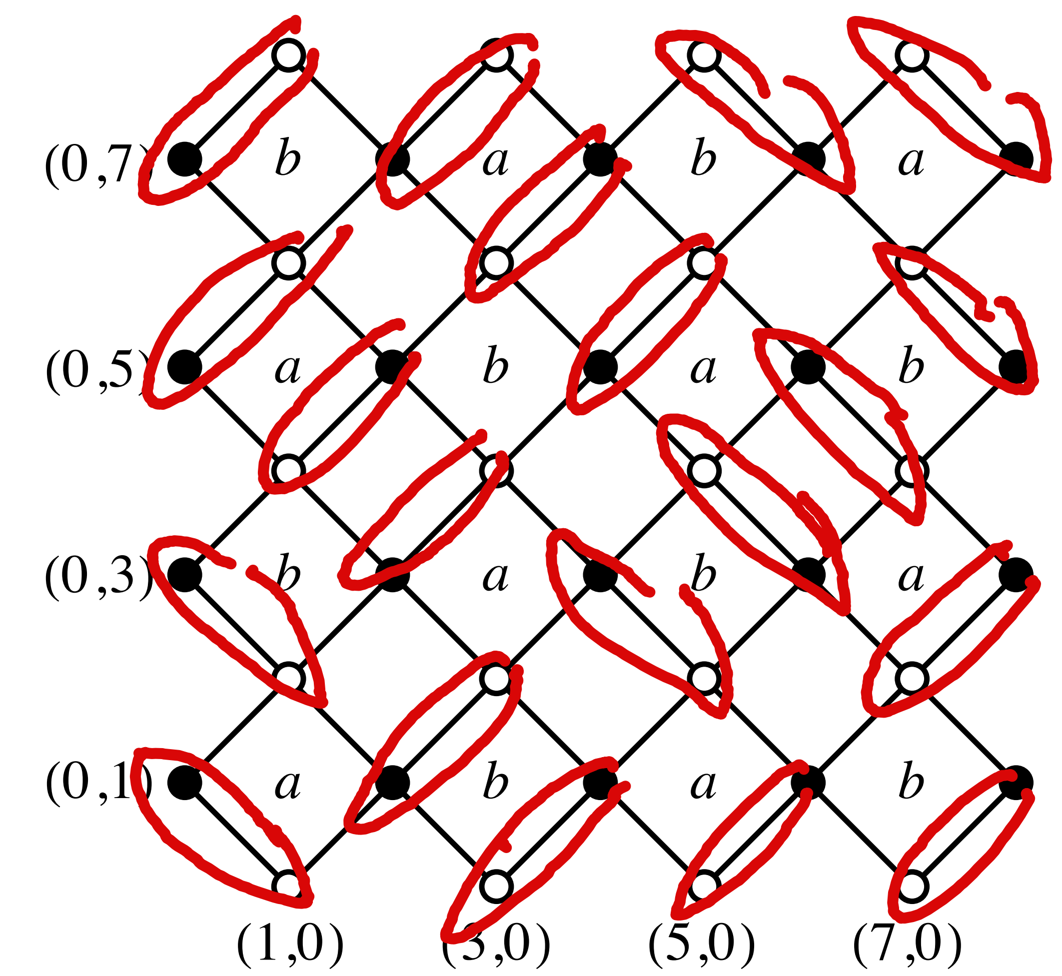
In particular, what is the probabilistic behavior found here

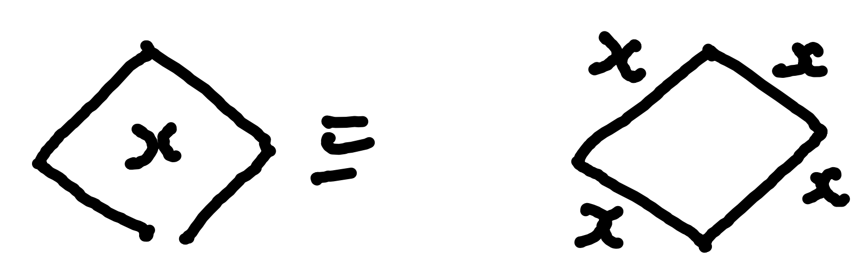
<u>Today</u>: Defining the paths and the ideas needed to show convergence of the top path to the Airy process (w. Duncan Dauvergne & Tom Finn)

Definitions



Aztec diamond
size 4





Dimer - Edge

Dimer Covering - A subset of edges covered exactly once by a dimer

[Can thicken dimers → dominoes]

[domino tiling]

Dimer Model: Let $w(M)$ be the product of the edge weights of M.

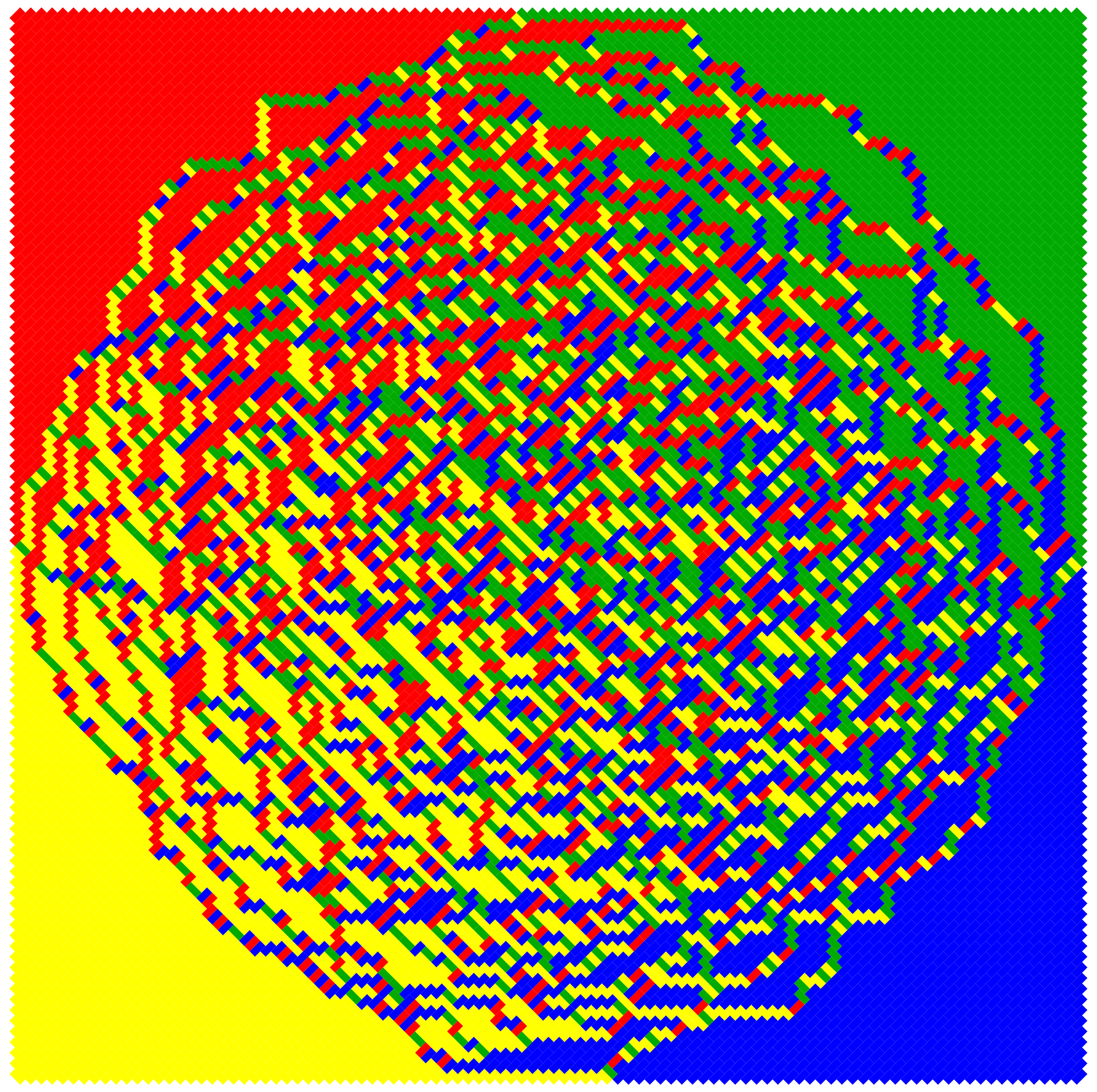
Then $\mathbb{P}[M] = \dfrac{w(M)}{Z}$ where Z is the partition function

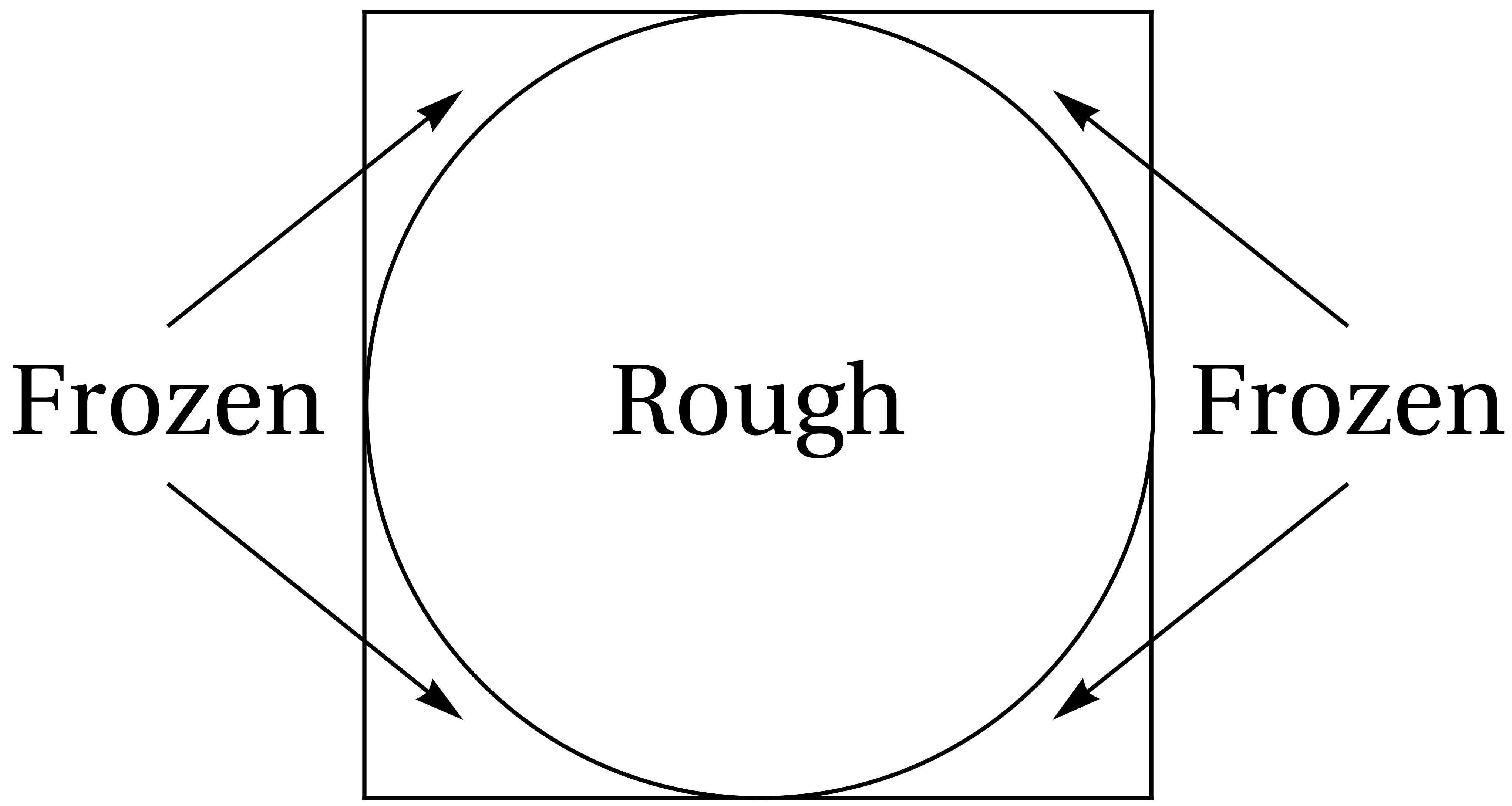
[Above $w(M) = a^{10} b^{10}$] Z is <u>known</u>.

Take $b=1$ (w.l.o.g)

The case $a=1$ (uniformly random tilings) and general tilings

- Number of tilings $= 2^{n(n+1)/2}$ Elkies-Kuperberg-Larsen-Propp 92
- Exact Sampling algorithm Propp 03
- Limit Shape Phenomena: Variational Principle Cohn-Kenyon-Propp 96
  (General tiling models) Conformal Structures and Geometry classification Astala-Duse-Prause-Zhong 20

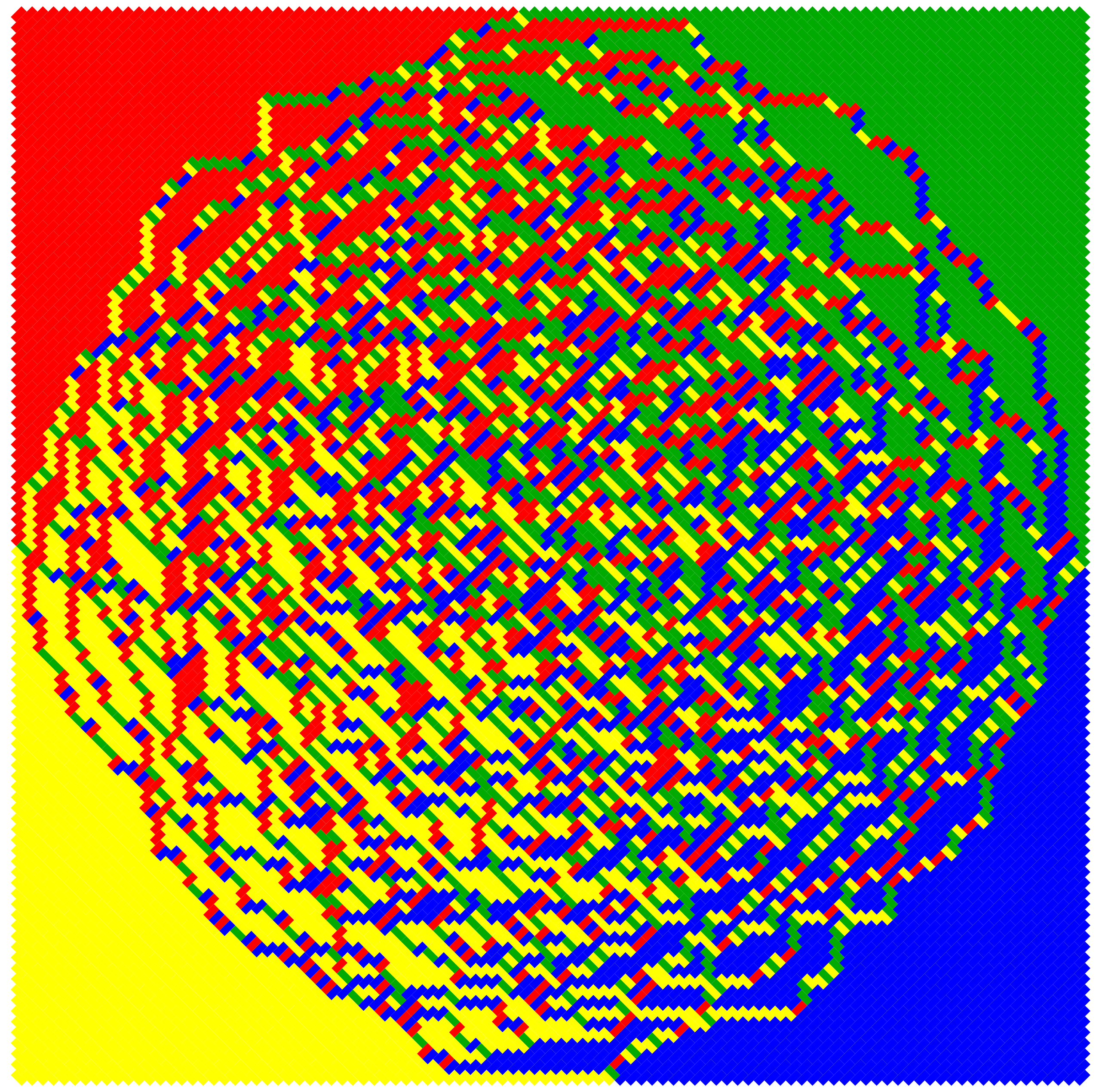




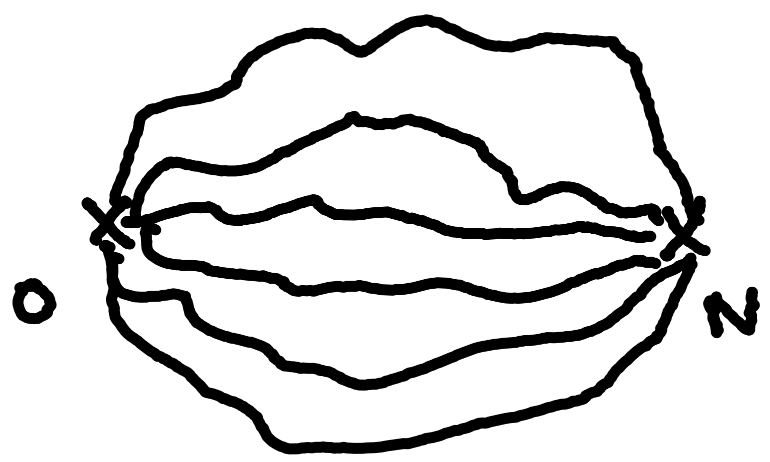
- Boundary Behavior: Uniform Aztec diamond Johansson 03
  General boundary conditions lozenge tilings Huang 21, Aggarwal-Huang 21

At the tangency points Aggarwal-Gorin 21 (lozenges)

Boundary behaviour

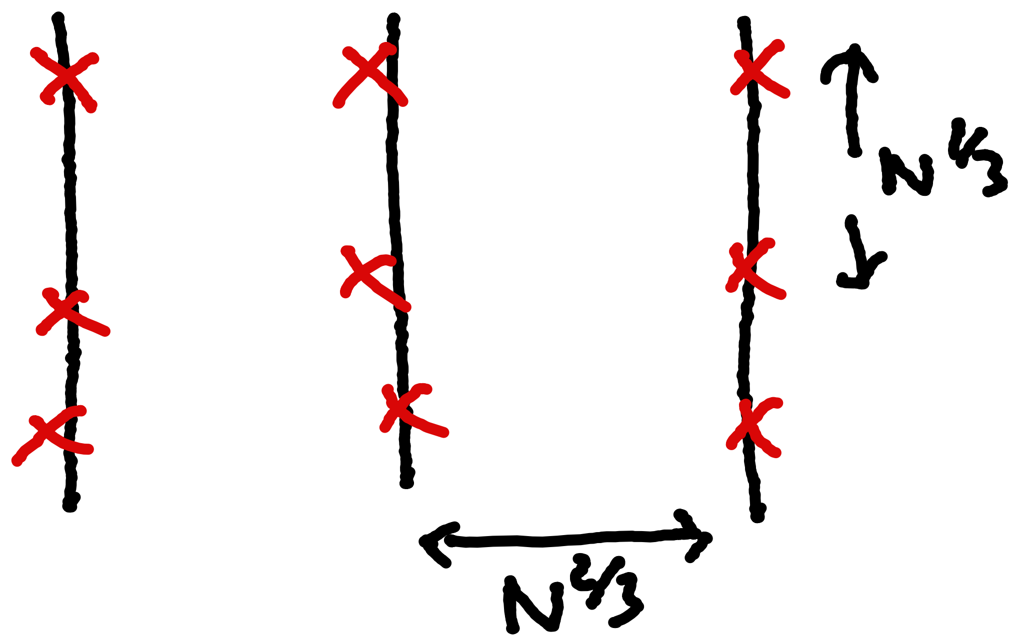


Airy Process :



Dyson BM – 1D independent Brownian Bridges conditioned not to intersect

After suitable centering rescaling : (at the top)



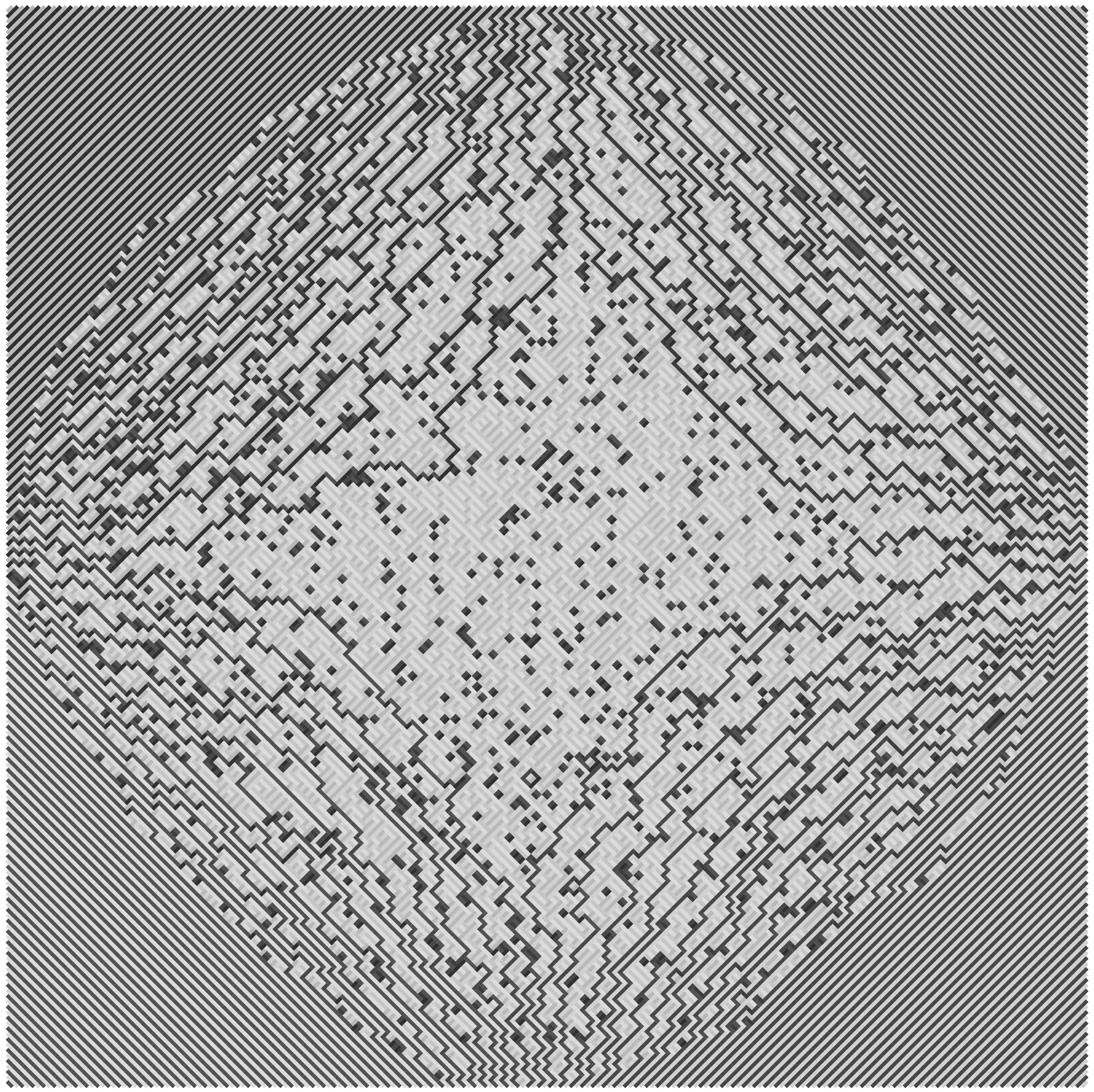
The point process converges to a determinantal point process called the extended Airy kernel point process.

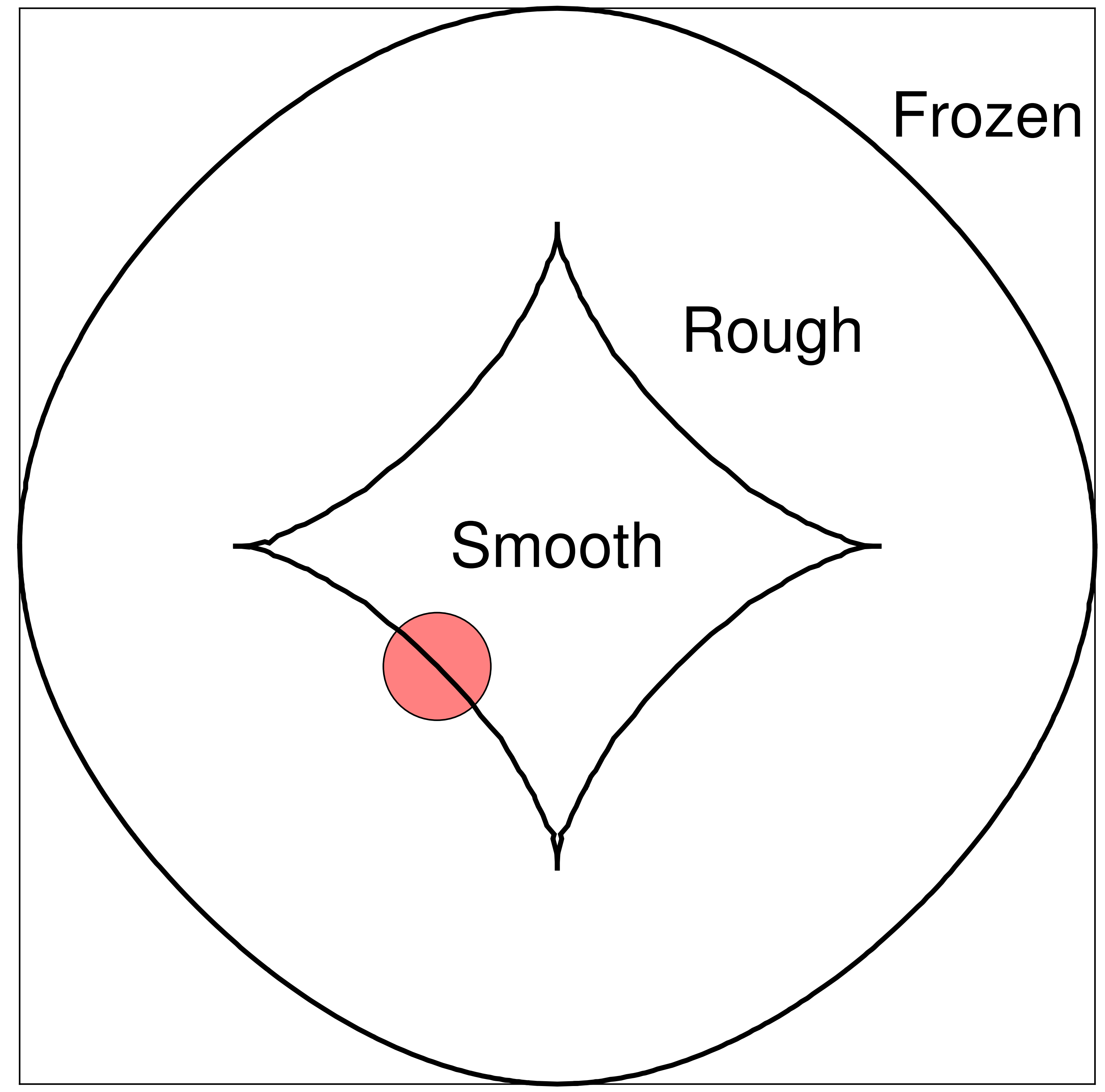
The last path converges to the Airy process.

Johansson 03 : Last path converges to the Airy process in the uniform Aztec diamond.

<u>Motivation</u>: Kenyon-Okounkov-Sheffield 03 characterized three possible macroscopic phases – what is the behaviour at the rough-smooth boundary ?

4m





8-degree curve

Frozen Deterministic

Rough Polynomial Decay (Delocalized)

Smooth Exponential (Localized)

Taken $a = \frac{1}{2}$. Dark - a-edges
Light - 1-edges

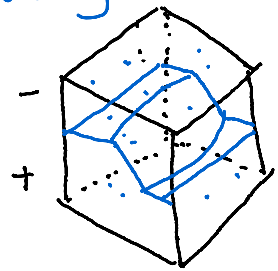
Taking $n = 4m$ [Here $m = 50$]

<u>Follow up Qu:</u> Is the rough-smooth boundary still an Airy process?
How is the boundary defined?

Other Models with a rough-smooth boundary

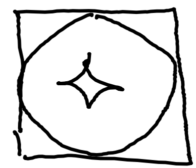
- 3D Ising model with sloped interface at low temperature

  Low temperature - small bubbles of opposite spins.



[Zero-temperature - No bubbles]

- Six-vertex model with Domain wall boundary condition with $\Delta < -1$



All unstudied....

- Three-periodic Lozenge tilings

Airy process believed to govern fluctuations at the interface of these models

<u>Main difficulties</u>
- No formulas (Airy process is defined using a Fredholm determinant)
- No close models that have previously been studied.
- Boundary may not be locally defined like for the frozen-rough boundaries or other KPZ universality class models.

Recent Progress

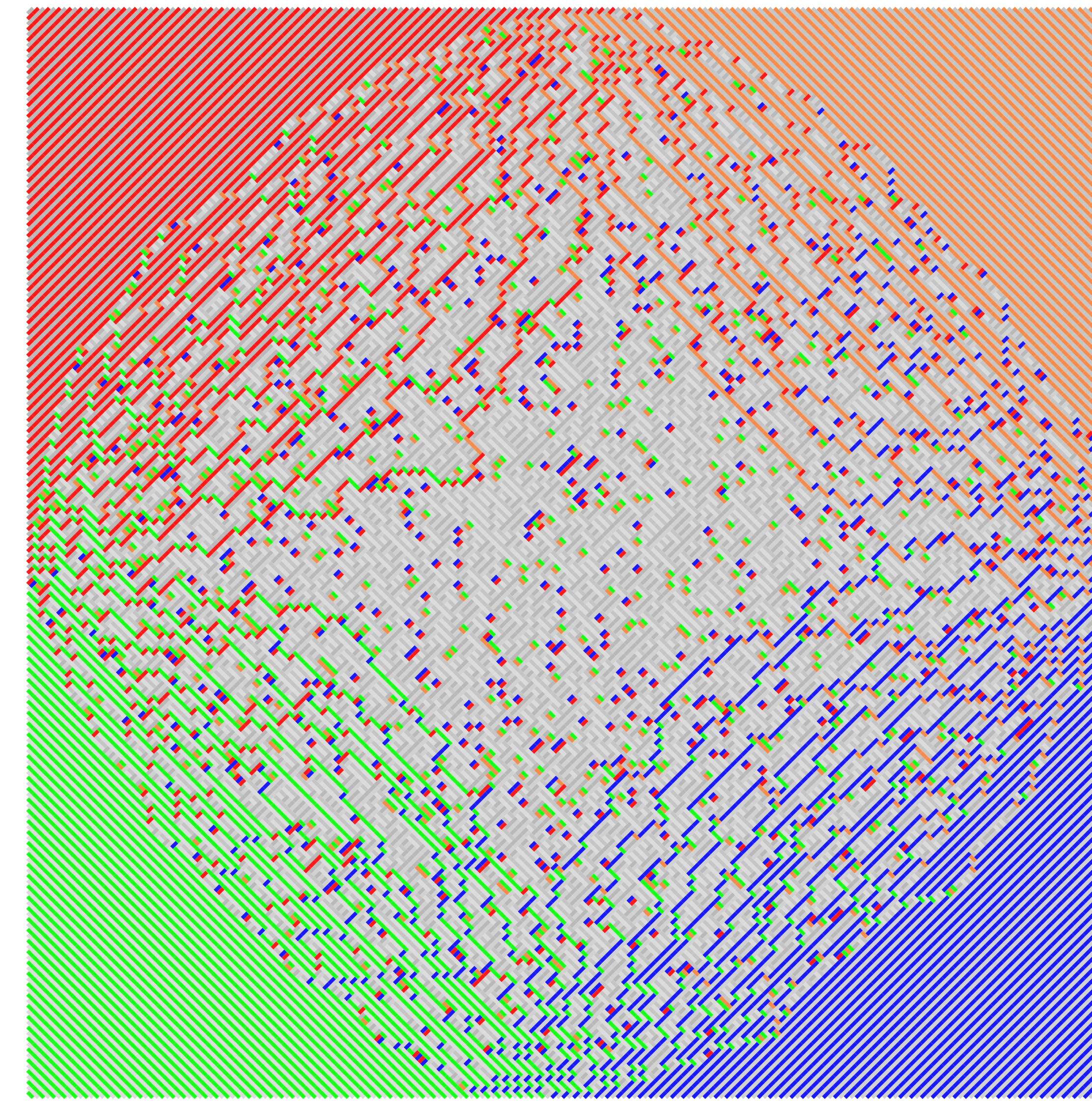
Correlation Kernels (block determinantal p.p., not Schur processes)

C.-Young 13

Duits-Kuijlaars 17

Berggren-Duits 19

C.-Duits 22



① Inverse K (Domino Shuffle)

② Matrix orthogonal polynomials

③ Wiener Hopf factorization (matrix symbols)

①↔③ are equivalent.

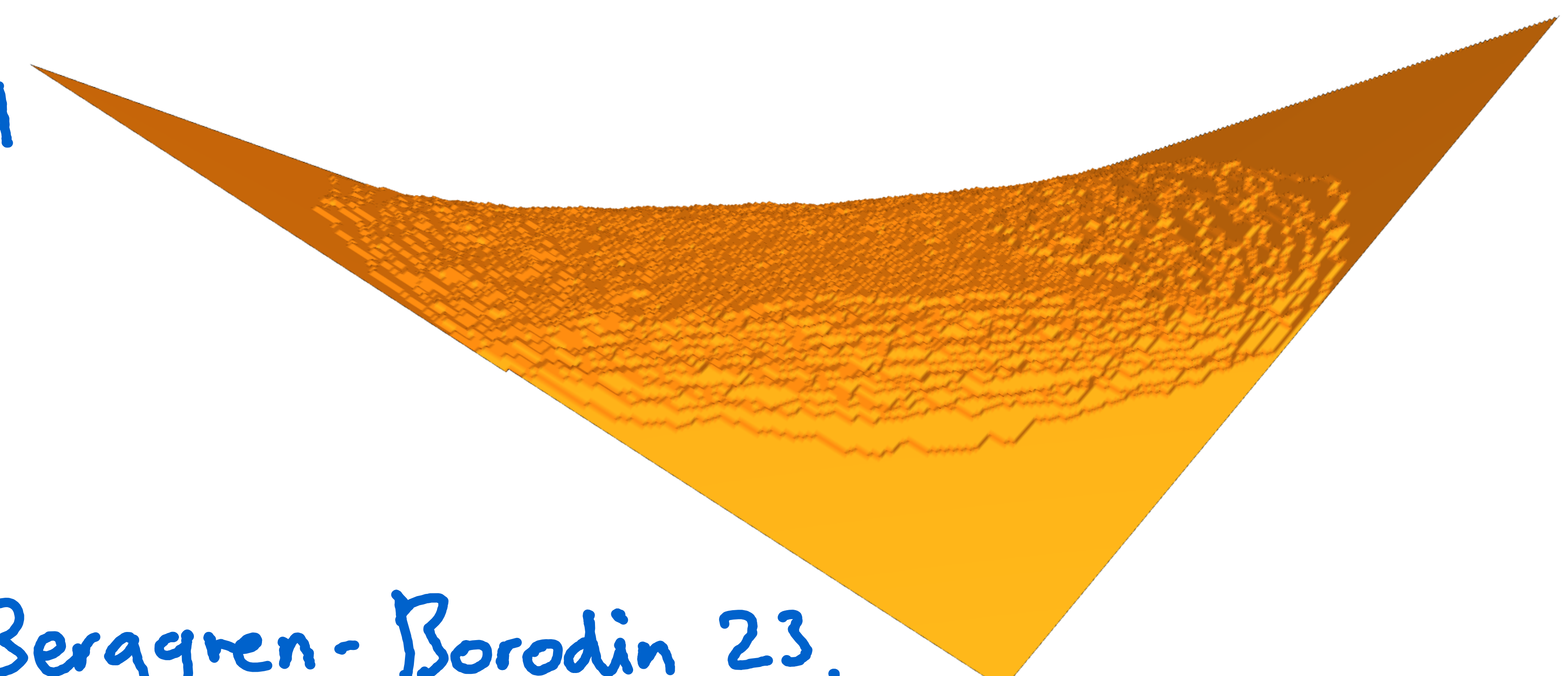
Asymptotics of the Dominoes:

C.-Johansson 14

Duits-Kuijlaars 17

Johansson-Mason 21

Bain 22



Generalizations:

Berggren 19

Borodin-Duits 22; Berggren-Borodin 23.

Rough-Smooth boundary:

Beffara-C.-Johansson 16, 20

Johansson Mason 23

Today: Johansson-Mason's holds for $a \to 0$ as $n \to \infty$.

We give the ideas needed to generalize this.

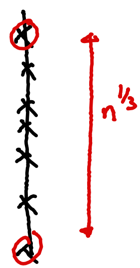
Behaviour of the dominoes at the rough-smooth boundary

C - Johansson 14

$$\text{“} \; K = K_{\text{smooth}} + n^{-1/3} K_{\text{Airy}} \; \text{”}$$

- K: Correlation kernel of det.p.p
- K_{smooth}: What's expected in the smooth phase
- $n^{-1/3} K_{\text{Airy}}$: Term associated from the Airy process

Picture :



[Recall a point process is determinantal if $\rho(x_1, \dots x_n) = \det\limits_{1 \le i,j \le n} K(x_i, x_j)$

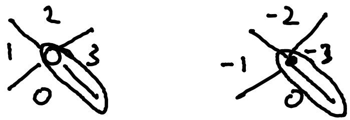
where K is called the correlation kernel]

In fact, the behaviour is <u>more</u> nuanced. The two terms are mixed and combined in nontrivial ways.

[K_{smooth} becomes the Gaussian part of the extended Airy kernel point process]

Removing the smooth parts

Associate to each tiling there is a height function

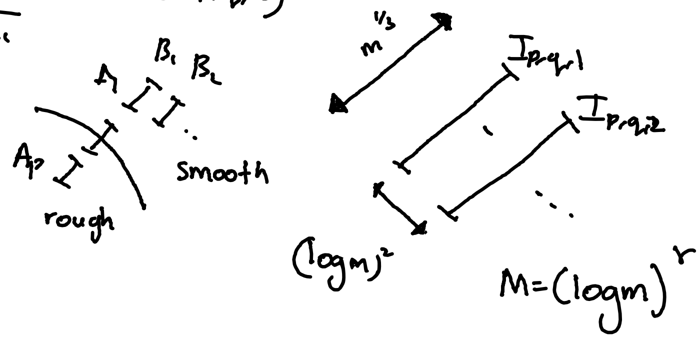


Around each white vertex, increase height by +1 clockwise, unless crossing a dimer, then drop by 3.

At the rough smooth boundary, introduce

$$N_M(A_p \times \{B_q\}) = \frac{1}{M}\sum_{k=1}^{M} \Delta h(I_{p,q,k})$$

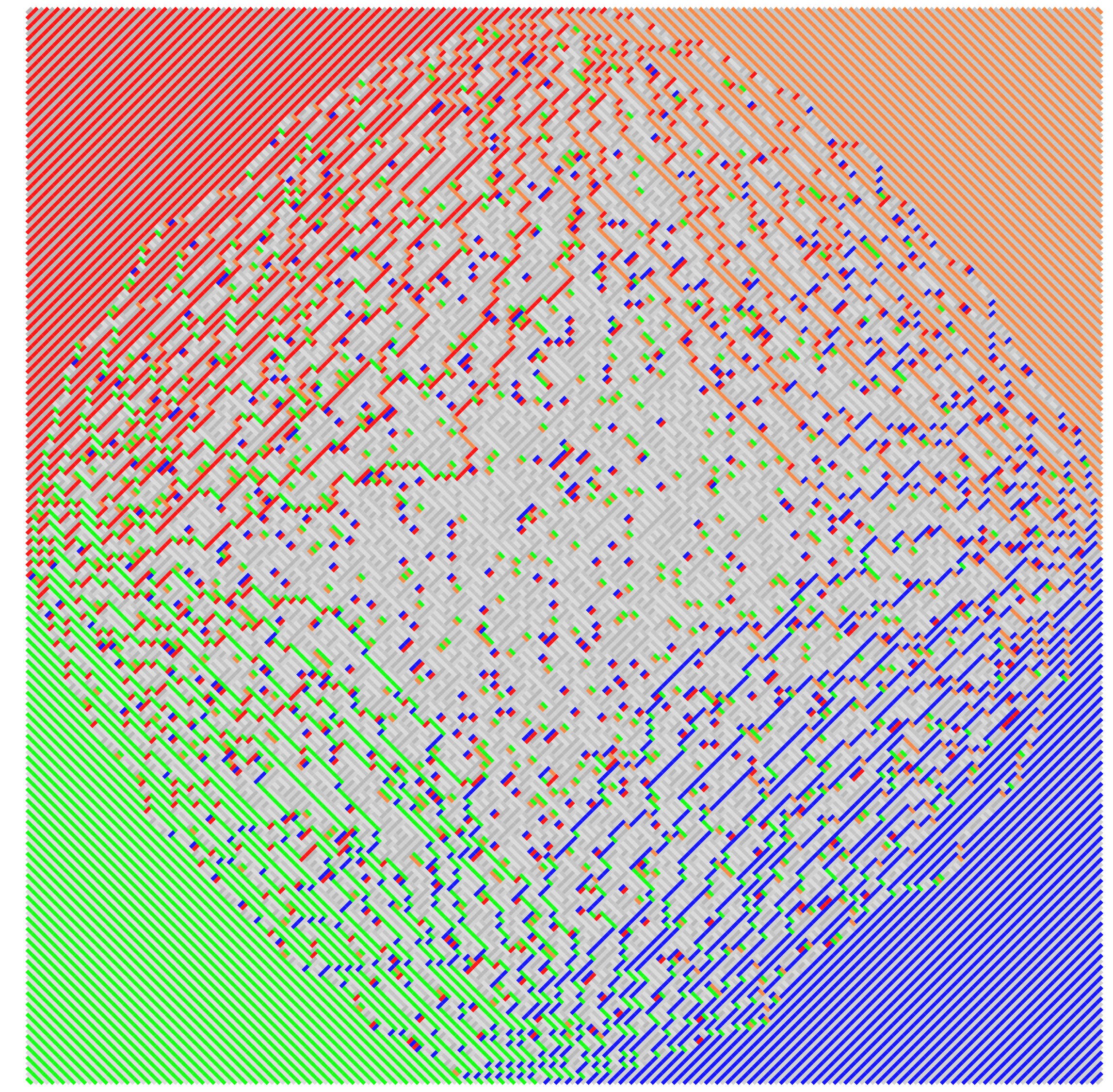
A_p: interval in $\mathbb{R}$; $\{B_q\}$: line



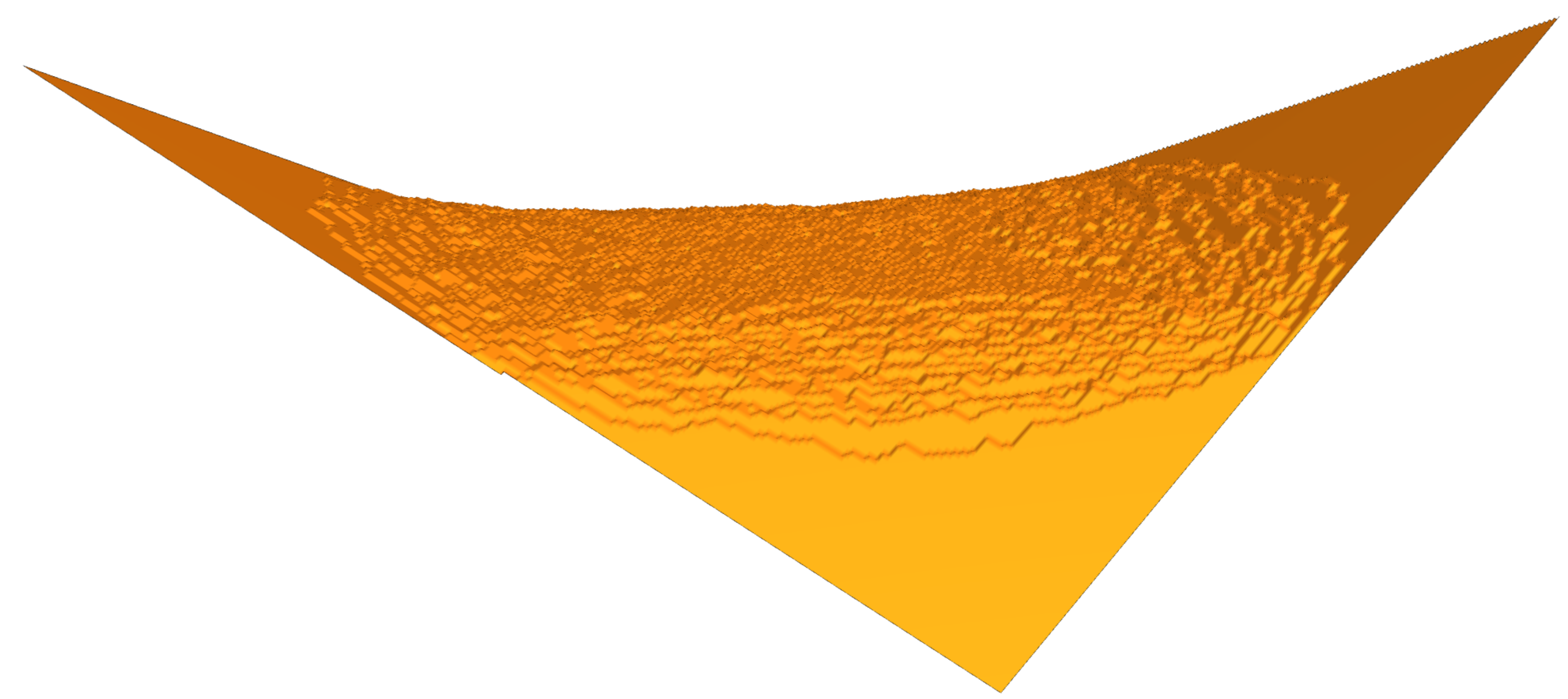
Beffara-C-Johansson 16: $N_M \Rightarrow$ Extended Airy kernel point process.

[Recall that we introduced this using Dyson Brownian Motion]

What's happening



- The smooth phase in the two-periodic Aztec diamond is flat.
- The averaging of the height function separates K_{smooth} & K_{Airy} as they are at different scales.

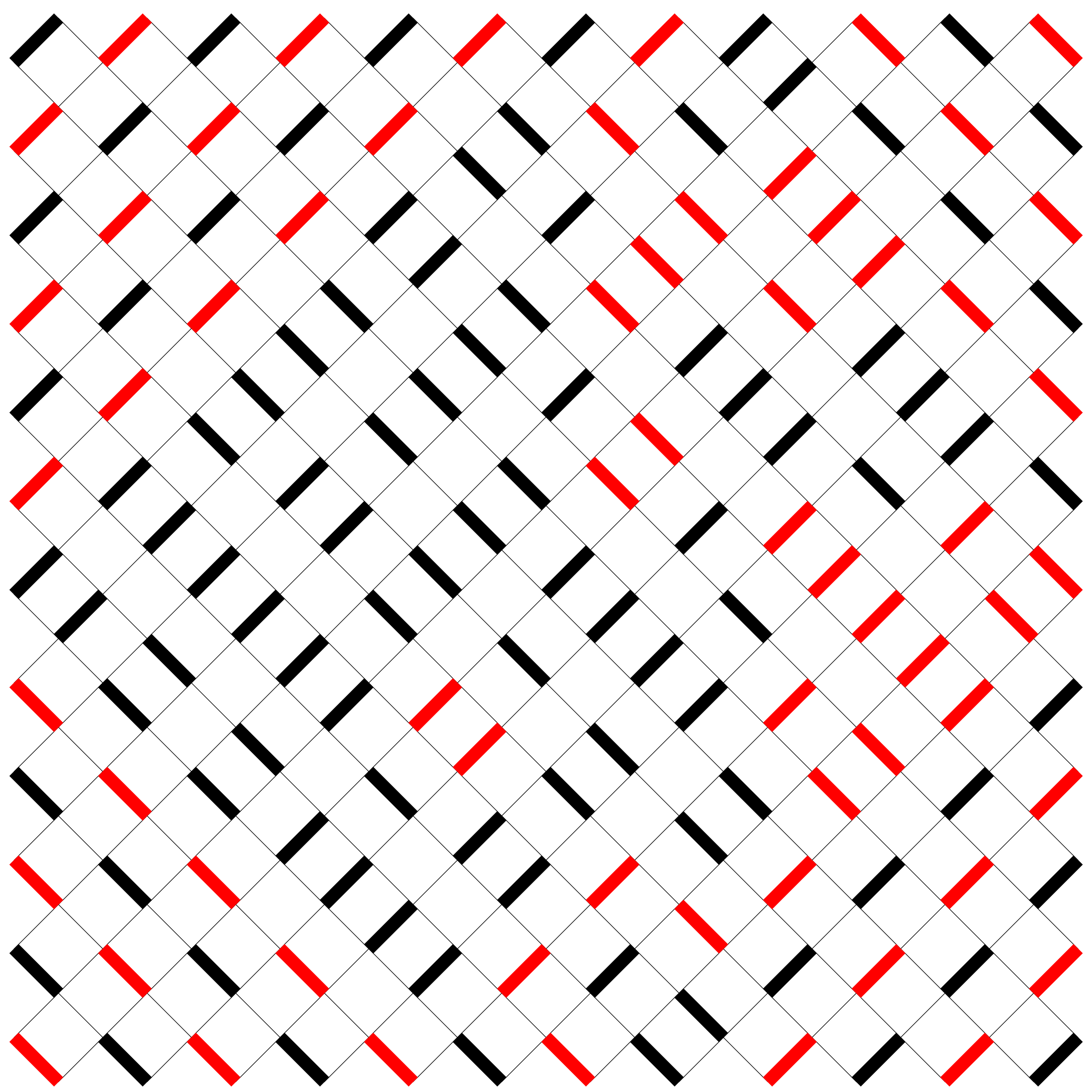


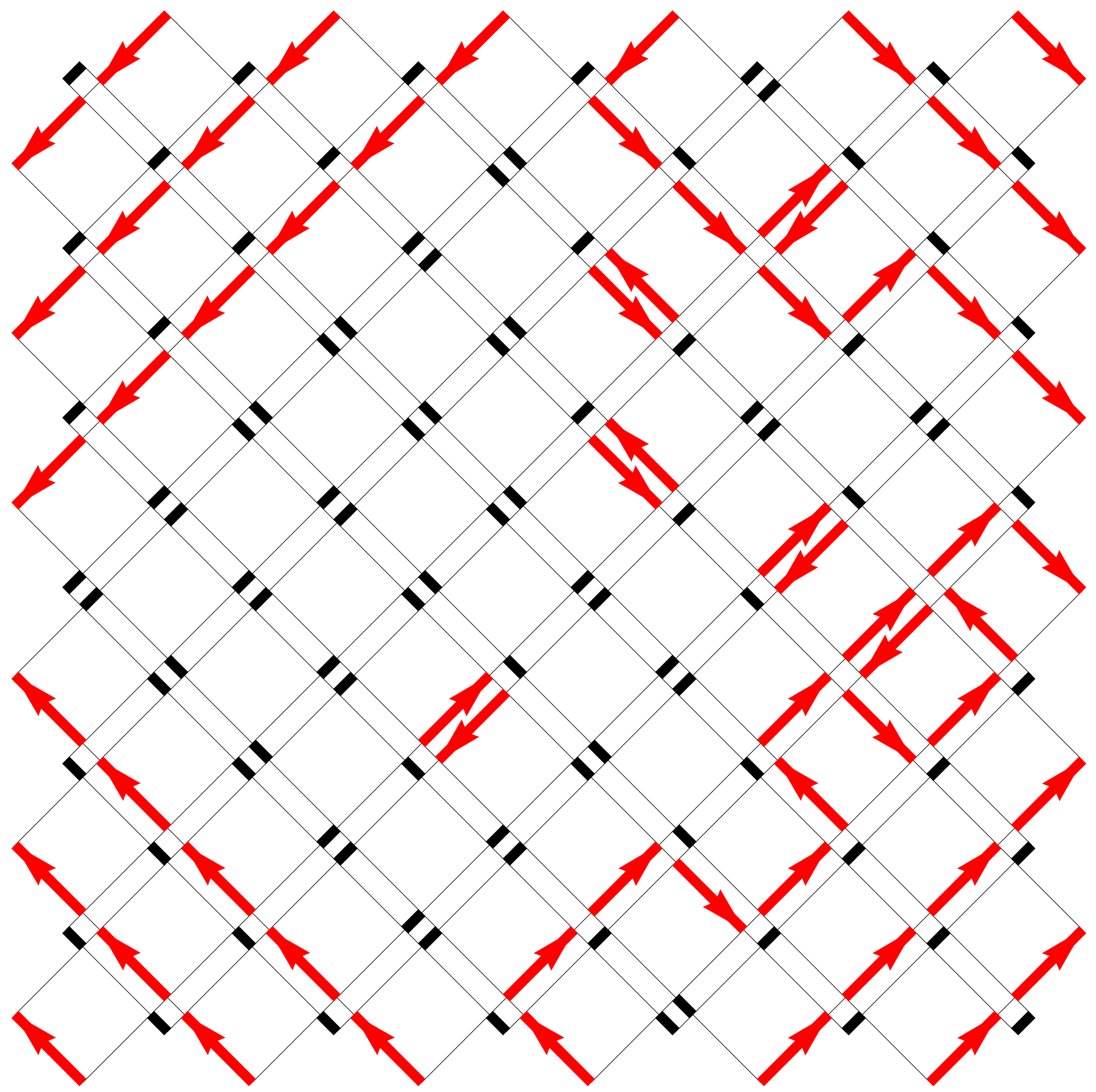
- Can we say more,

$$\underbrace{N_m}_{\text{A random (signed) measure}} \Rightarrow \underbrace{\text{Extended Airy kernel point process}}_{\text{A point process}}$$

Are there paths which are captured by N_m?

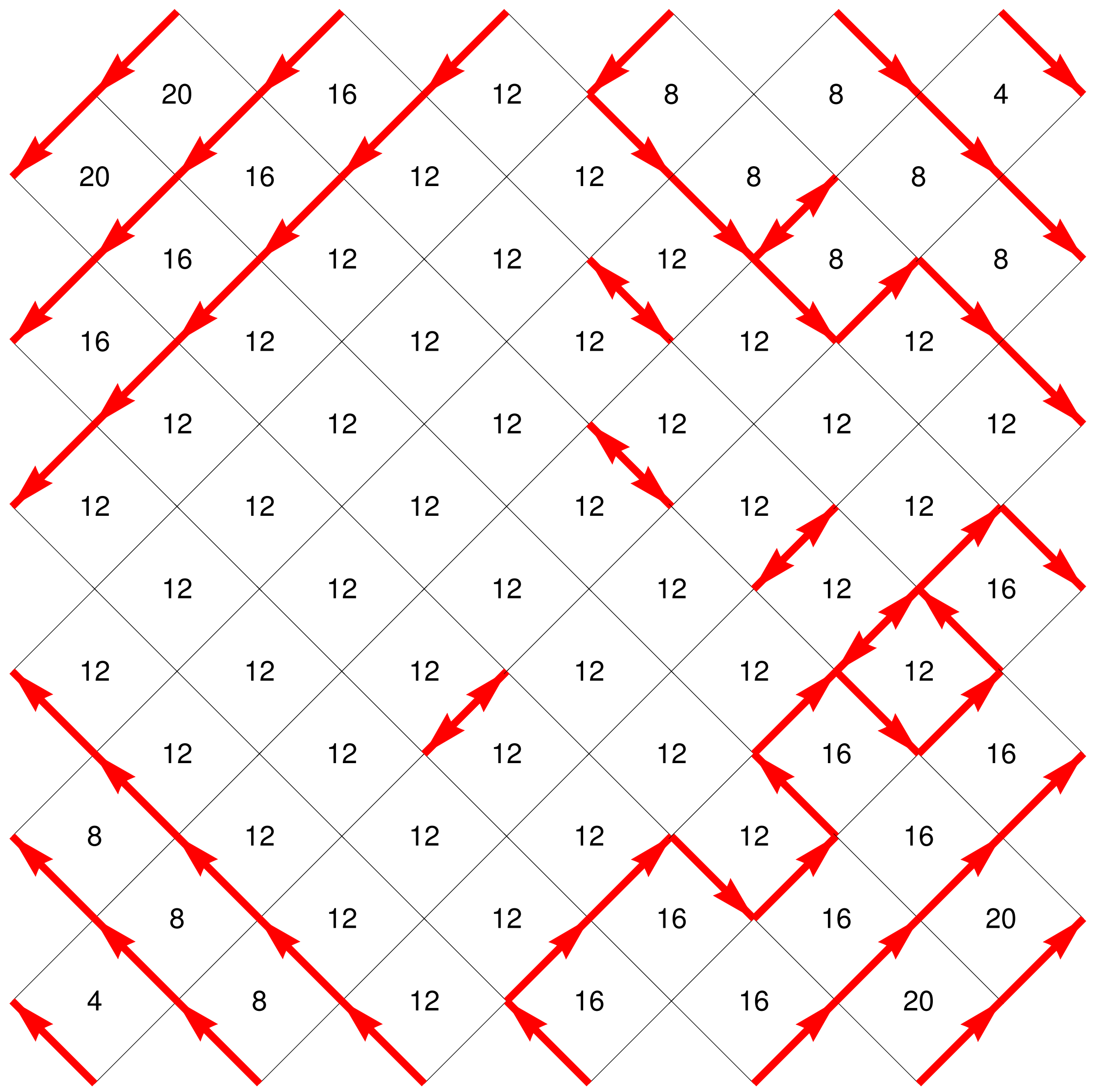
Squishing → Lines and loops





Contract the 1-faces.

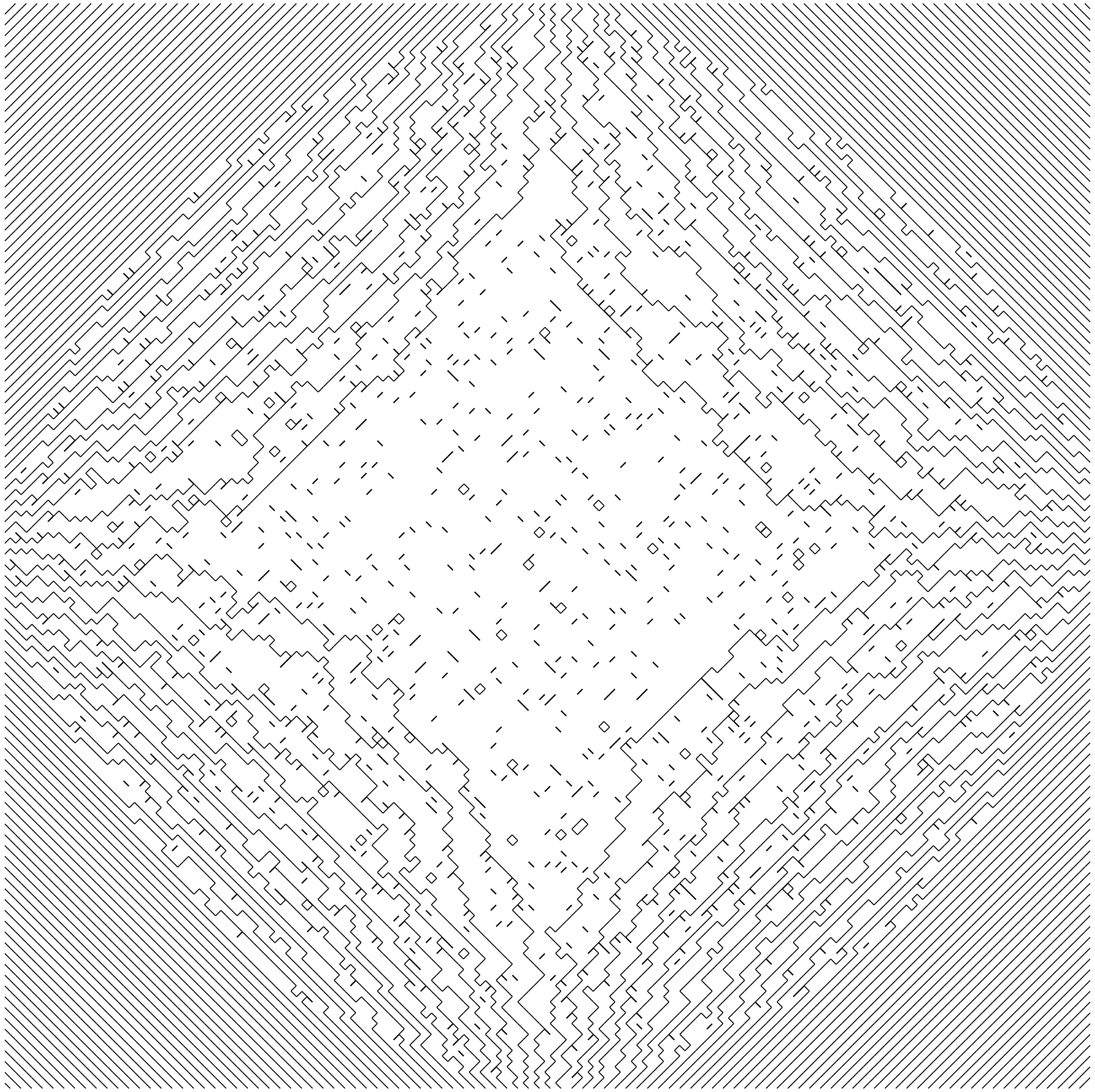
What remains is the a-faces (and dimers on a-edges)



Possible behaviour (up to a certain choice)

- loops
- lines connecting two boundaries (paths)
- double edges

These mark the level lines of the height function.



Beffara-C-Johansson 20: For $a < 1/3$, a measure defined by counting path crossings converges to the extended Airy kernel point process.

Johansson-Mason 23: For $a = m^{-1+\varepsilon}$ there is a last path converging to the Airy process.

Reasons for small a : * Peierls argument controlling the loops for $a < 1/3$

* Relatively few backtracks for $a \to 0$ as $m \to \infty$ (no need to have global control or local control of the paths).

Going further: All from Dauvergne-C-Finn (23+)

Lemma (Loop Control) For $0<a<1$ and for L large

$$\mathbb{P}_{Az}\left[\exists \text{ a loop } \gamma \text{ s.t } L(\gamma) > L\right] \leq C_1 a^{L}$$

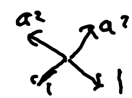
We can also replace $\mathbb{P}_{At}$ by any "flat" box or the full plane

An eventual consequence is

Dauvergne-C-Finn: For $a<1$, a measure that counts paths crossed converges to the extended Airy kernel point process.

Proof of Lemma: For a 'flat' box, rewrite using a drifted spanning tree (gauge equivalence)

The tree is generated by a biased random walk.



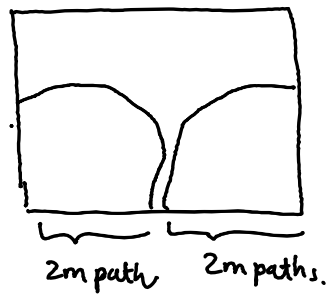
Informally, to generate a large loop, we require the random walk to make a large "excursion" against its drift.

Conditioning on the paths, using Wilson's algorithm we can generate the remaing dimer configuration. Final step is to spanning tree objects to squished paths.

Lemma Let x be a point in the smooth region. Then

$$\mathbb{E}_{A_{\pm}}[h(x)] = 2m \quad \text{and} \quad \mathrm{Var}_{A_{\pm}}(h(x)) < C$$

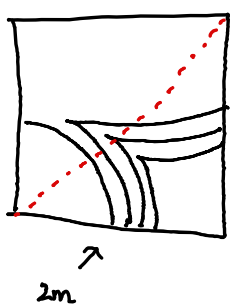
Consequence:



(By building another random measure)

BCJ-20 No squished path in the full-plane smooth phase a.s.

Global geometry: There could exist onion type events:



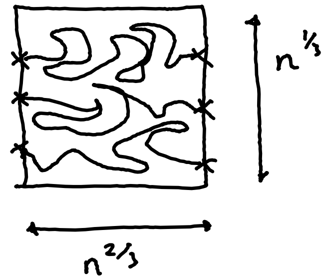
} paths have to be narrow.

Removed by resampling: gives a positive probability of macroscopic loops, providing a contradiction

Conclude that there is a last path providing the separation.

The last steps....

Paths locally can backtrack:



To control this, we use a Gibbs resampling property, which shows that in the limit the paths are onedimensional (regularity of the paths).

Hence we show that along the diagonal, the top path converges to the Airy process (in some sense).

Future Work

Finn-Macera-Shea   Cusp at the rough-smooth boundary.
(Pearcey kernel)

Thanks for listening!