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Arctic curves of the four-vertex model

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Based on arXiv:2307.03076 - joint work with:

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A.G. Pronko (Steklov Mathematical Institute, RAS, Saint Petersbourg)

The model

Motivation:

The four-vertex model is just a little bit more difficult than 'plain vanilla' dimers, but not too much... Definitely easier than:

- five-vertex model [DeGier-Kenyon-Watson'21][Kenyon-Prause'22];
- six-vertex model at $\Delta < 1$ [FC-Pronko'10][Aggarwal'20].

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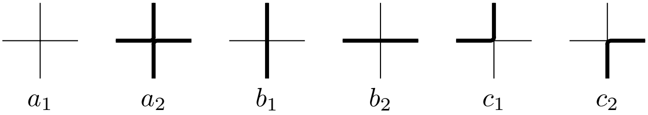

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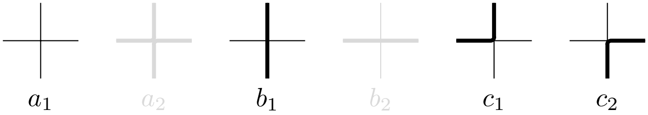

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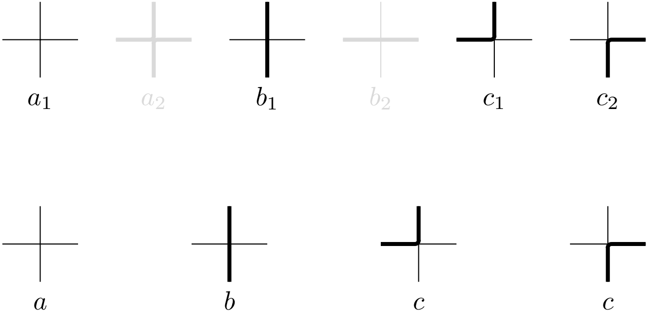

[Li-Park-Widom’90][Bogoliubov’07-’10]

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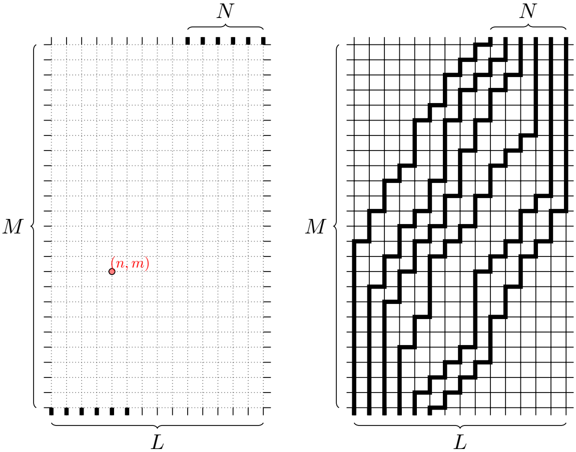
'scalar-product' boundary conditions



The model

$$Z_{L,M,N}(a,b,c) = \sum_{\{\text{conf}\}} a^{\#a} b^{\#b} c^{\#c}$$

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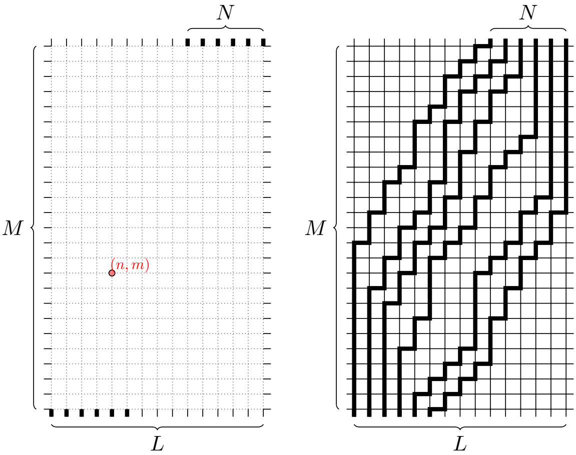

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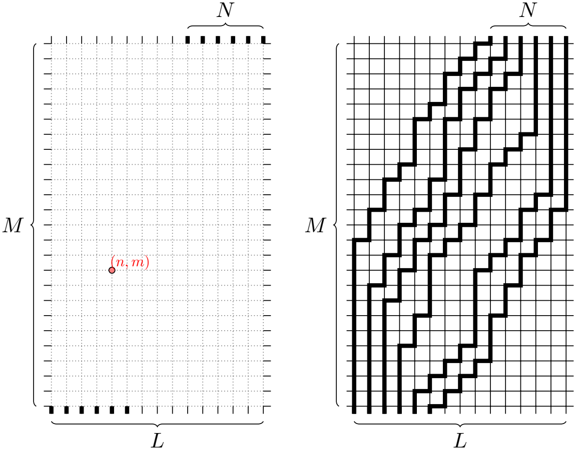

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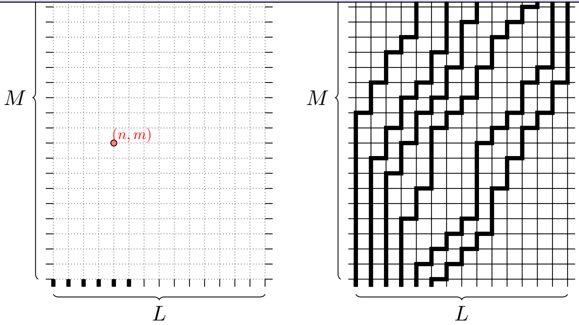

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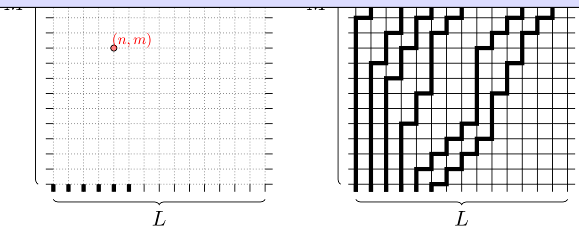
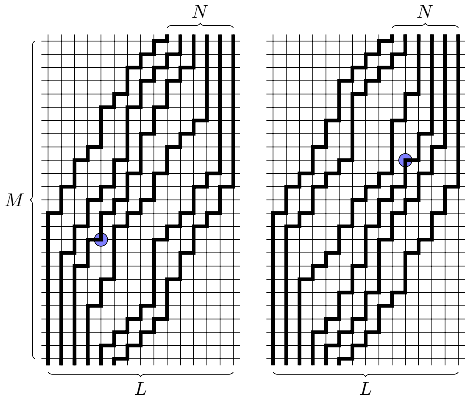
$$Z_{L,M,N} = \mathrm{PL}(N, L-N, M-L+1)$$



Some properties of the model

1) Reflection symmetry:



$$\Phi_{L,M,N}(L-n+1, M-m+1) = \Phi_{L,M,N}(n,m)$$

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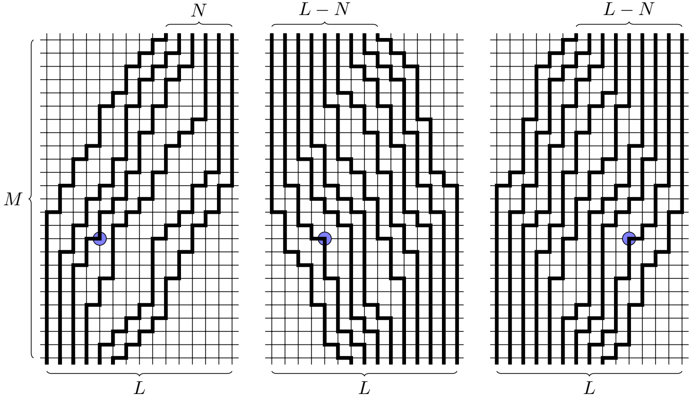

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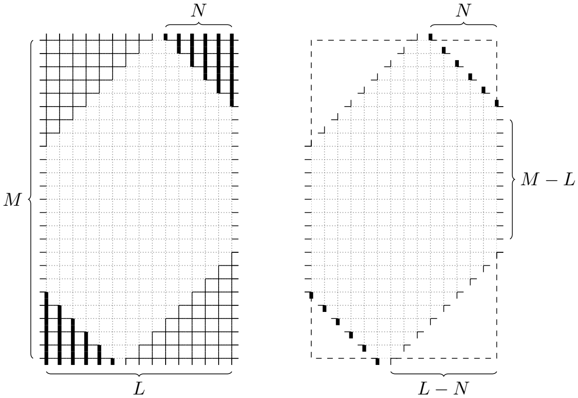
3) Equivalent hexagonal domain:



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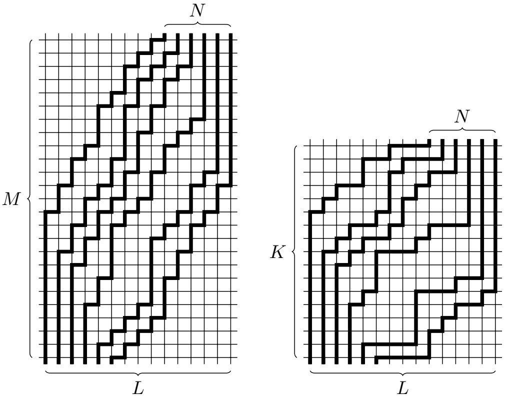
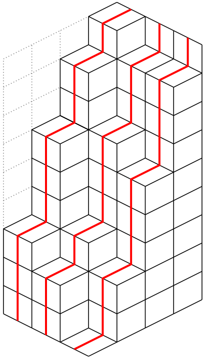
3) Equivalent hexagonal domain.

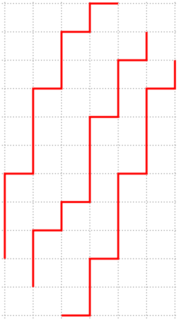
4) Four-vertex model and NILP: $K = M - L + N + 1$



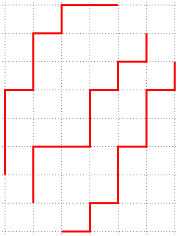
Lattice paths and plane partitions



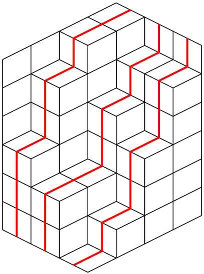
column-strict BPP
box of size
$N \times (L-N) \times (M-N)$



four-vertex model
N paths
on a $L \times M$ lattice



NILP
N paths
on a $L \times K$ lattice

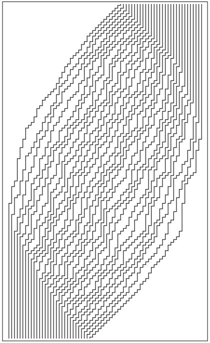


BPP
box of size
$N \times (L-N) \times (K-N)$

$K := M - L + N + 1$

Here $N = 3,\ L = 7,\ M = 12,\ K = 9$

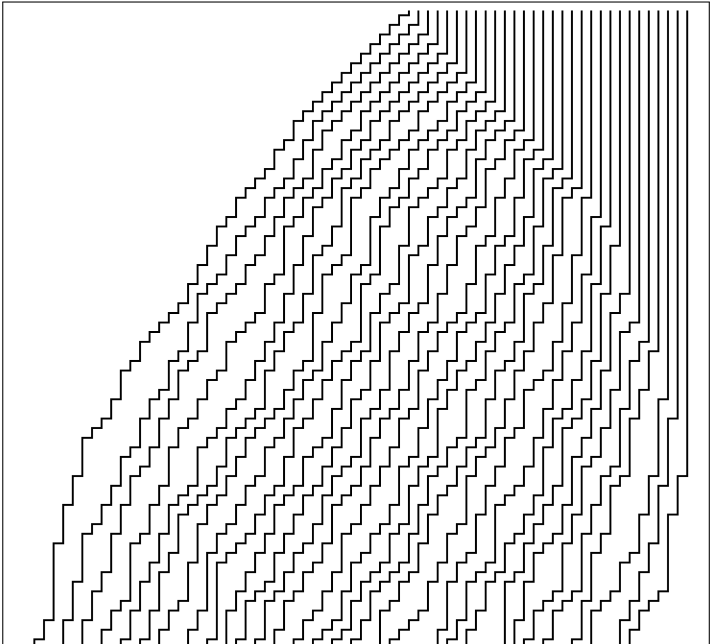
The problem



$(L, M, N) = (70, 120, 30)$

Uniformly sampled configuration, generated with CFTP [Propp-Wilson'96]

The problem



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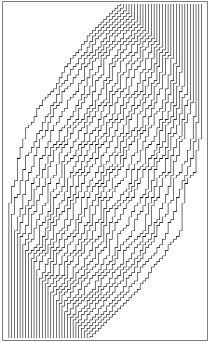

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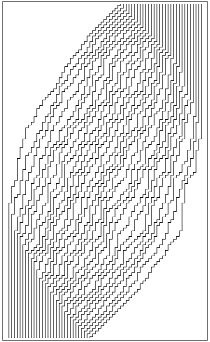

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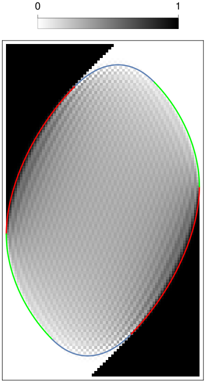

$(140, 240, 60)$
density of a-vertices
10^5 simulations

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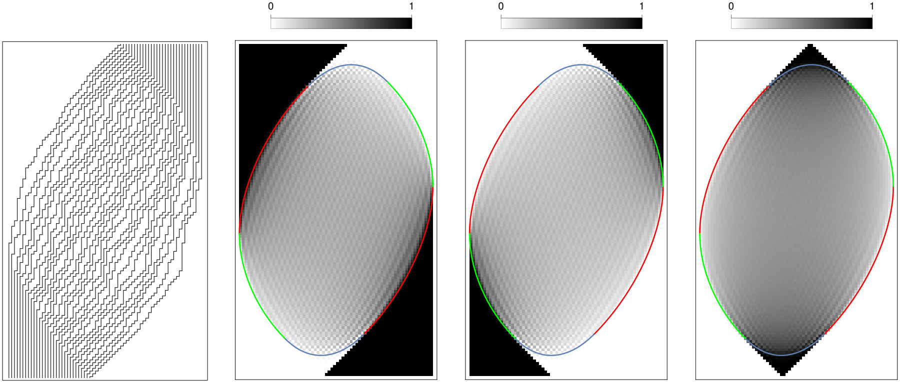

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$(140, 240, 60)$
density of b-vertices
10^5 simulations

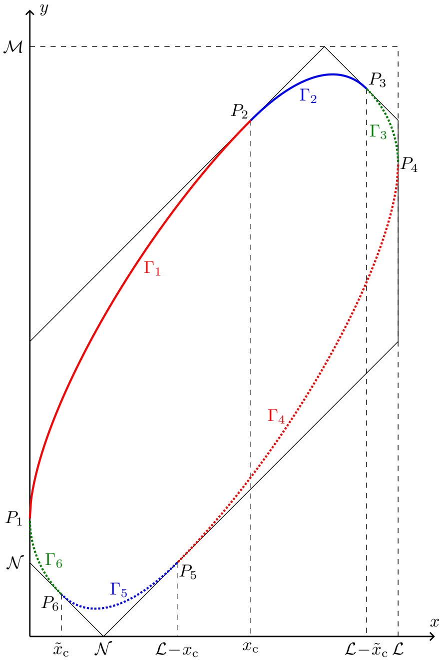
$(140, 240, 60)$
density of c-vertices
10^5 simulations

Uniformly sampled configurations, generated with CFTP [Propp-Wilson'96]

Main result

- lattice coordinates: $(n, m) \in [1, L] \times [1, M]$
- Scaling limit:

$$L = \lceil \mathcal{L}\ell \rceil, \quad M = \lceil \mathcal{M}\ell \rceil, \quad N = \lceil \mathcal{N}\ell \rceil, \quad n = \lceil x\ell \rceil, \quad m = \lceil y\ell \rceil, \quad \ell \to \infty$$



$\mathcal{N} : \mathcal{L} : \mathcal{M} = 1 : 5 : 8$

Main result

Theorem. [BCMP'23] The portions Γ_1 and Γ_2 of the arctic curve of the four-vertex model with 'scalar-product' boundary conditions are given by:

$$\begin{cases} \Gamma_1: \quad y = f_1(\mathcal{L},\mathcal{M},\mathcal{N};x), & x \in (0, x_c], \\ \Gamma_2: \quad y = f_2(\mathcal{L},\mathcal{M},\mathcal{N};x), & x \in [x_c, \mathcal{L} - \tilde{x}_c], \end{cases}$$

where

$$f_1(\mathcal{L},\mathcal{M},\mathcal{N};x) = \frac{\mathcal{M}\mathcal{N}(\mathcal{L}-2x) + (\mathcal{M}+\mathcal{N})\mathcal{L}x}{\mathcal{L}^2} + 2\frac{\sqrt{\mathcal{M}\mathcal{N}(\mathcal{L}-\mathcal{N})(\mathcal{M}-\mathcal{L})(\mathcal{L}-x)x}}{\mathcal{L}^2},$$

$$f_2(\mathcal{L},\mathcal{M},\mathcal{N};x) = (\mathcal{L}-\mathcal{M}-\mathcal{N}-x) + 2f_1(\mathcal{L},\mathcal{M},\mathcal{N};x),$$

and

$$x_c = \frac{(\mathcal{M}-\mathcal{L})(\mathcal{L}-\mathcal{N})}{\mathcal{M}-\mathcal{L}+\mathcal{N}}, \qquad \tilde{x}_c = \frac{(\mathcal{M}-\mathcal{L})\mathcal{N}}{\mathcal{M}-\mathcal{N}}.$$

Fluctuations of Γ_1 and Γ_2 are governed by the Tracy-Widom distribution.

Remarks

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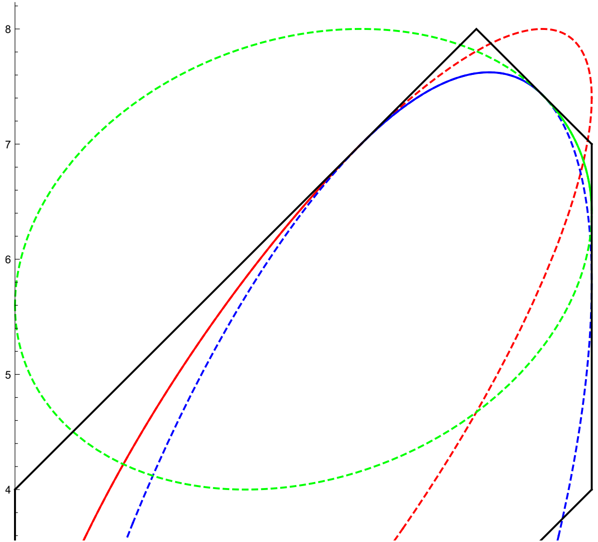

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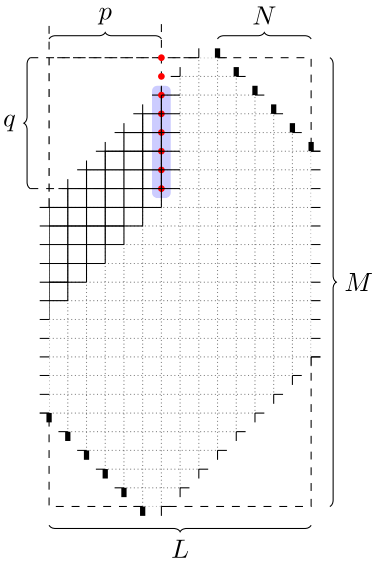
- The fluctuations of the arctic curve in the present model provides one more example in support of the universality of Tracy-Widom distribution.
- Two possible derivations of the above result:
  - Tangent Method [FC-Sportiello'16]: very intuitive and efficient, but still heuristic;
  - EFP Method: slightly more involved, but, at least in the present case, it may be carried out in full rigour. And gives also the Tracy-Widom fluctuations for free.

Emptiness Formation Probability (EFP)



$A_{p,q} := \{q \text{ top vertices in } p^{th} \text{ vert. line}\}$

$B := \{\text{vertices in the hex. domain}\}$

$\Theta := A_{p,q} \cap B$

$|\Theta| = p + q - L + N =: \tilde{q}$

$F_{L,M,N}(p,q) := \frac{\#\text{configs: } (v{=}a,\ \forall v{\in}\Theta)}{\#\text{configs}}$

$p + q > L - N$

$p \leq L - N$

$p + q < M - N$

Emptiness Formation Probability (EFP)

- If the (p, q) topleft rectangle is relatively small then the probability $F_{L.M.N}(p, q)$ is close to one.
- The probability $F_{L.M.N}(p, q)$ is a decreasing function of p and q , and vanishes if these are deep enough into the disordered region.
- In the scaling limit, $F_{L.M.N}(p, q) \to 1$ outside the arctic curve, and $F_{L,M,N}(p, q) \to 0$ as soon as (p, q) penetrates the disordered region.
- In other words, in the scaling limit, $F_{L,M,N}(p, q)$ has a stepwise behaviour, from 1 to 0, in correspondence of the arctic curve.

Hahn log-gas

Hahn measure:

$$w_n^{(\alpha,\beta)}(x) = \binom{\alpha + x}{x}\binom{\beta + n - x}{n - x}, \qquad x \in [0, n]$$

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$$w_n^{(\alpha,\beta)}(x) = \binom{\alpha + x}{x}\binom{\beta + n - x}{n - x}, \qquad x \in [0, n]$$

Provided that $\alpha, \beta > -1$ or $\alpha, \beta < -n$, we may define

$$\left\{Q_{k,n}^{(\alpha,\beta)}\right\}_{k=0}^{n} \quad \text{such that} \quad \sum_{x=0}^{n} w_n^{(\alpha,\beta)}(x) Q_{k,n}^{(\alpha,\beta)}(x) Q_{l,n}^{(\alpha,\beta)}(x) = \delta_{k,l}.$$

We have

$$Q_{k,n}^{(\alpha,\beta)}(x) = (-1)^k \sqrt{\binom{n}{k} \frac{n!(2k+\alpha+\beta+1)(\alpha+1)_k(\alpha+\beta+1)_k}{(\alpha+\beta+1)_{n+1}(\beta+1)_k(n+\alpha+\beta+2)_k}} \times {}_3F_2\left(\begin{matrix} -k,\ k+\alpha+\beta+1,\ -x \\ \alpha+1,\ -n \end{matrix}\middle| 1\right)$$

known as (normalized) Hahn polynomials [Koekoek-Lesky-Swarttouw'10].

Hahn log-gas

Hahn measure:

$$w_n^{(\alpha,\beta)}(x) = \binom{\alpha + x}{x}\binom{\beta + n - x}{n - x}, \qquad x \in [0, n]$$

Let $\mathbf{x} := \{x_1, \ldots, x_n\}$, with $0 \leq x_1 < \cdots < x_s \leq n$,
$=$ ordered set of positions of s particles on the discrete interval $[0, n]$

Probability measure on $[0, n]^s$:

$$P_{n,s}^{(\alpha,\beta)}[\mathbf{x}] = \frac{1}{Z(\alpha, \beta, s, n)} \prod_{1 \leq i < j \leq s} (x_i - x_j)^2 \prod_{i=1}^{s} w_n^{(\alpha,\beta)}(x_j)$$

The normalization constant

$$Z(\alpha, \beta, s, n) = \sum_{0 \leq \mathbf{x} \leq n} \prod_{1 \leq i < j \leq s} (x_i - x_j)^2 \prod_{i=1}^{s} w_n^{(\alpha,\beta)}(x_j)$$

is the partition function of the Hahn log-gas ($\alpha, \beta > -1$ or $\alpha, \beta < -n$).

Hahn log-gas

- The partition function as an Hankel determinant [Szegö'39]:

$$Z(\alpha, \beta, s, n) = \frac{1}{n!} \det_{1 \le i,j \le s} \left[\sum_{x=0}^{n} x^{i+j-2} w_n^{(\alpha,\beta)}(x) \right]$$

built from the the moments of Hahn measure.

Hahn log-gas

Let:

$$H(d, \alpha, \beta, s, n) := \sum_{0 \leq \mathbf{x} \leq d} P_{n,s}^{(\alpha,\beta)}[\mathbf{x}]$$

This is nothing but the 'gap probability', i.e., the probability of having, for the Hahn log-gas with parameters α, β, n, and s particles, no particle with coordinate larger that d.

- In the context of N NILP on the $L \times K$ lattice (or plane partitions, or lozenge tilings), the Emptiness Formation Probability at (p, q) evaluates to $H(K - q, p - N, L - N - p, N, K)$ [Johansson'00].

Hahn log-gas

- The 'gap probability' as an Hankel determinant [Szegö'39]:

$$H(d,\alpha,\beta,s,n) = \frac{1}{n!}\det_{1\le i,j\le s}\left[\sum_{x=0}^{d} x^{i+j-2} w_n^{(\alpha,\beta)}(x)\right]$$

- The 'gap probability' as a Fredholm determinant [Gaudin-Mehta'60s]:

$$H(d,\alpha,\beta,s,n) = \det\left[1 - K_{n,s}|_{(d,n]}\right],$$

where $K_{n,s}|_{(d,n]}$ is a discrete integral operator acting on $L^2(d,n]$ with kernel

$$K_{n,s}(x,y) = \sum_{k=0}^{s} Q_{k,n}^{(\alpha,\beta)}(x) Q_{k,n}^{(\alpha,\beta)}(y)\sqrt{w_n^{(\alpha,\beta)}(x) w_n^{(\alpha,\beta)}(y)}, \quad x,y \in [0,n],$$

i.e., the Christoffel-Darboux kernel for (normalized) Hahn polynomials.

Representation for EFP

Proposition. [BCMP'23] The Emptiness Formation Probability in the four-vertex model with N lines on the $L \times M$ lattice may be written as:

$$F_{L,M,N}(p,q) = H(d,\alpha,\beta,s,n)$$

with parameters

$$\begin{aligned}
d =& M - N + \min(p,N) - p - q \\
\alpha =& |N - p| \\
\beta =& L - N - p \\
s =& \min(p,N) \\
n =& M - L + \min(p,N).
\end{aligned}$$

- The conditions $\alpha, \beta > -1$ are evidently satisfied.
- The evaluation is based on the bijection between the four-vertex model and NILP.

Arctic curve

Asymptotic behaviour of $F^{(p,q)}_{L,M,N}$ in the scaling limit

$$\Updownarrow$$

Behaviour of $H(d,\alpha,\beta,s,n)$ in the limit $\ell \to \infty$, where:

$$d = \lfloor d_0 \ell \rfloor, \quad \alpha = \lfloor \alpha_0 \ell \rfloor, \quad \beta = \lfloor \beta_0 \ell \rfloor, \quad s = \lceil s_0 \ell \rceil, \quad n = \lceil n_0 \ell \rceil,$$

with $\alpha_0, \beta > 0$, and $s_0 < d_0 < n_0$.

- Inspired by Random Matrix models, one would rescale $x_j = \lfloor \mu_j \ell \rfloor$, interpret the sums as Riemann sums and, in the large ℓ limit, replace them with integrals. Correspondingly, one would introduce a density $\rho(\mu)$, which may be evaluated by solving some variational problem, etc ...

Arctic curve

Heuristically, $H(d, \alpha, \beta, s, n) \sim \frac{\int_0^{d_0} \rho(\mu) \mathrm{d}\mu}{\int_0^{n_0} \rho(\mu) \mathrm{d}\mu} \sim \Theta(d_0 - R_0)$,

where $R_0 = R_0(\alpha_0, \beta_0, s_0, n_0)$ is the right endpoint of the support of $\rho(\mu)$.

Thus the arctic curve is given by:

$$R_0(\alpha_0, \beta_0, s_0, n_0) = d_0,$$

see [Johansson'00] for a rigorous derivation.

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- We do not need full knowledge of $\rho(\mu)$, but only its support.
- This would be anyway some piece of work, but fortunately, already solved! [Baik-Kriecherbauer-McLaughlin-Miller'07]

Arctic curve

Heuristically, $H(d, \alpha, \beta, s, n) \sim \frac{\int_0^{d_0} \rho(\mu) \mathrm{d}\mu}{\int_0^{n_0} \rho(\mu) \mathrm{d}\mu} \sim \Theta(d_0 - R_0)$,

where $R_0 = R_0(\alpha_0, \beta_0, s_0, n_0)$ is the right endpoint of the support of $\rho(\mu)$.

Thus the arctic curve is given by:

$$R_0(\alpha_0, \beta_0, s_0, n_0) = d_0,$$

see [Johansson'00] for a rigorous derivation.

- We do not need full knowledge of $\rho(\mu)$, but only its support.
- This would be anyway some piece of work, but fortunately, already solved! [Baik-Kriecherbauer-McLaughlin-Miller'07]
- in our notations,

$$R(\alpha_0, \beta_0, s_0, n_0) = \\ = \left(\frac{\sqrt{(s_0 + \alpha_0 + \beta_0)(s_0 + \alpha_0)(n_0 - s_0)} + \sqrt{(s_0 + \alpha_0 + \beta_0 + n_0)(s_0 + \beta_0) s_0}}{(2 s_0 + \alpha_0 + \beta_0)} \right)^2 .$$

Fluctuations

Choosing some suitable value of p, we may write for EFP:

$$F_{L,M,N}(p,q) = H(M-p-q, p-N, L-N-p, N, M-L+N),$$

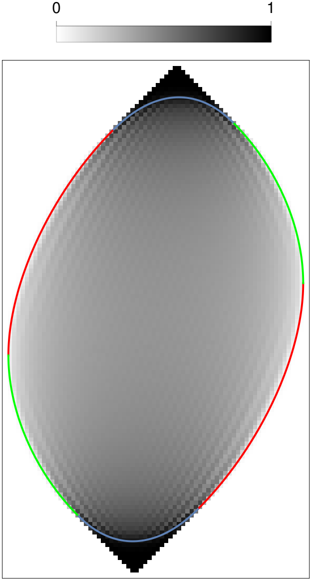
Let ξ denote the value of the topmost thick edge in the p^{th} column.
It follows form the definition of EFP, and from its Fredholm determinant representation that

$$\begin{aligned}\mathbb{P}(\xi < M-q) &= F_{L,M,N}(p,q) \\ &= \det[1 - K_{M-L+N,N}|_{(M-p-q,M-L+N]}]\end{aligned}$$

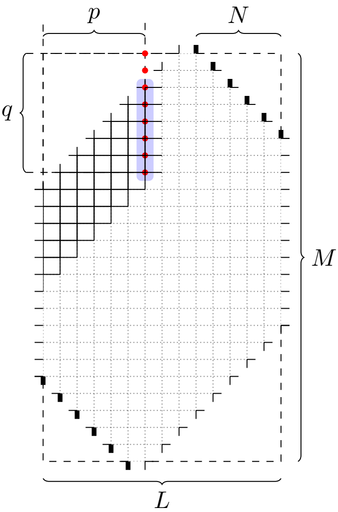
Focus now on values of $M-q$ in the vicinity of the arctic curve, $p + \ell R_0$.
In such regime, in the scaling limit, the Christoffel-Darboux kernel for Hahn measure tends to the Airy kernel [Baik-Kriecherbauer-McLauglin-Miller'07].
In our model and notations, we have, for suitable constant t:

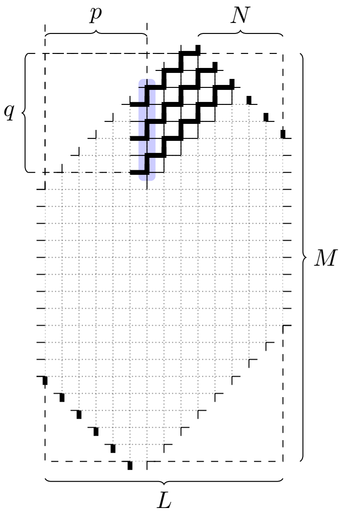
$$\lim_{\ell\to\infty} \mathbb{P}\left(\frac{\xi - p - \ell R(\alpha_0, \beta_0, s_0, n_0)}{(t\ell)^{1/3}} \leq x\right) = \det[1 - A|_{(x,\infty)}].$$

EFP and AFP



EFP and AFP





Emptiness Formation Probability
$F_{L,M,N}(p,q)$

Anti-ferroelectric phase
Formation Probability
$G_{L,M,N}(p,q)$

Representation for AFP

Proposition. [BCMP'23] The Anti-ferroelectric Formation Probability in the four-vertex model with N lines on the $L \times M$ lattice may be written as:

$$G_{L,M,N}(p,q) = H(d, \alpha, \beta, s, n)$$

with parameters

$$\begin{aligned} d =& L - N + \min(\tilde{r}, M - L + 1) - 2 - q + \tilde{r} \\ \alpha =& |M - L - \tilde{r} + 1| \\ \beta =& N - \tilde{r} \\ s =& \min(\tilde{r}, M - L + 1) \\ n =& L - N + \min(\tilde{r}, M - L + 1) - 1. \end{aligned}$$

And then proceed as above to evaluate Γ_2, and recover Tracy-Widom for fluctuations. Next, use symmetries of the model to get $\Gamma_3, \ldots, \Gamma_6$.

Some open questions

- Limit shapes? We tried, but probably not hard enough.
  Maybe the approach of [Kenyon-Prause'20] could be useful?
  In case, are fluctuations of the limit shape again governed by GFF.
- Fluctuations of configurations near a contact point?
  In ASMs and lozenge tilings, GUE corner process [Gorin'14].
  But here reflection symmetry is broken; has this any effect?