# Move-reduced graphs on the torus

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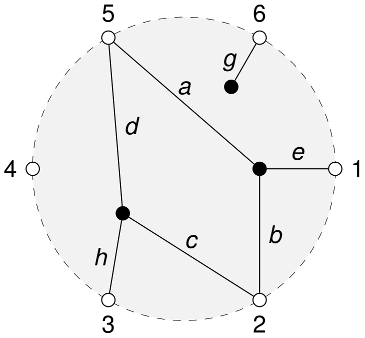
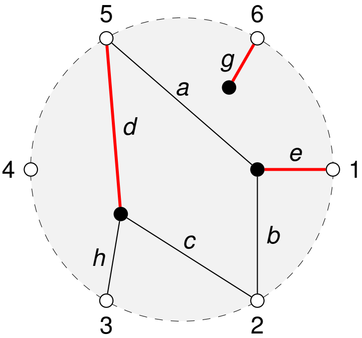
# Dimer model on the disk




bipartite graph with weight : edges $\to \mathbb{R}_{>0}$

$n := \#$boundary white vertices

# Dimer covers




boundary $= 234$

weight $= deg$

- A **dimer cover** $M$ is a subset of the edges that uses each internal vertex exactly once and each boundary vertex at most once.
- The **boundary** $\partial M$ of $M$ is the subset of boundary white vertices not used by $M$.

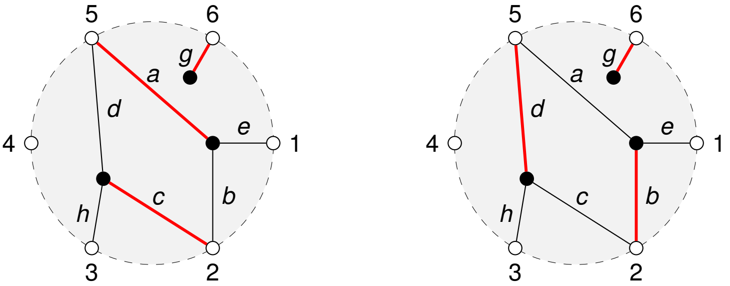
$$k := \#\partial M = \#\text{white vertices} - \#\text{black vertices}.$$

- The **weight** of $M$ is the product of weights of edges in $M$.

# Dimer partition functions

- For $I$ a $k$-element subset of $\{1, 2, \ldots, n\}$, the **dimer partition function** is defined as

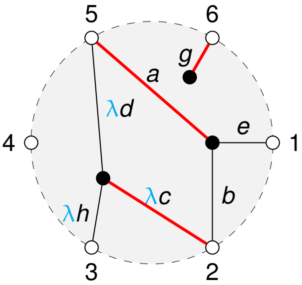
$$Z_I := \sum_{M : \partial M = I} \text{weight}(M).$$




$$Z_{134} = acg + bdg$$

[Postnikov, 2006] boundary measurement : weight $\mapsto (Z_I)$

# Gauge transformations




$$Z_{134} = \lambda acg + \lambda bdg$$

Multiplying the weights of all edges incident to an internal vertex by a constant $\lambda \in \mathbb{R}_{>0}$ rescales each $Z_I$ by $\lambda$

# Boundary measurement map [Postnikov, 2006]

- We identify weighted bipartite graphs if they are related by gauge transformations. So, we need to identify $(Z_I)$ and $(\lambda Z_I)$.
- Projective space $\mathbb{RP}^{\binom{n}{k}-1} := \mathbb{R}^{\binom{n}{k}} - \{0\}/\text{scaling}$.

$$\text{boundary measurement} : \text{weight}/\text{gauge} \mapsto (Z_I)/\text{scaling} \subset \mathbb{RP}^{\binom{n}{k}-1}.$$

**Question:** Which points in projective space arise from dimer models?

[Postnikov, 2006] The **totally nonnegative Grassmannian** $\mathrm{Gr}_{\geqslant 0}(k, n)$.

# The totally nonnegative Grassmannian

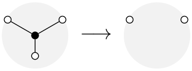
- The Grassmannian $\mathrm{Gr}(k,n)$ is the space of $k$-dimensional subspaces of $\mathbb{R}^n$.
- We can represent a point $V \in \mathrm{Gr}(k,n)$ as the rowspan of a $k \times n$ matrix $M$ of full rank.
- The $k \times k$ minors of $M$ are (homogeneous) coordinates on $\mathrm{Gr}(k,n)$, called **Plücker coordinates**. Let $\Delta_I$ denote the Plücker coordinate using columns indexed by $I$, where $I$ is a $k$-element subset of $\{1, 2, \ldots, n\}$. They give an embedding of $\mathrm{Gr}(k,n)$ in $\mathbb{RP}^{\binom{n}{k}-1}$.
- If $V = \text{rowspan} \begin{bmatrix} 1 & a & 0 & -c \\ 0 & b & 1 & d \end{bmatrix}$, then the Plücker coordinates are

$$\Delta_{12} = b, \Delta_{13} = 1, \Delta_{14} = d, \Delta_{23} = a, \Delta_{24} = ad + bc, \Delta_{34} = c.$$

- The subset $\mathrm{Gr}_{\geqslant 0}(k,n)$ of $\mathrm{Gr}(k,n)$ where all Plücker coordinates are positive nonnegative is called the totally nonnegative Grassmannian. If $a, b, c, d \geqslant 0$, then $V \in \mathrm{Gr}_{\geqslant 0}(2,4)$.

# Reduction moves [Postnikov, 2006]


Parallel edge reduction

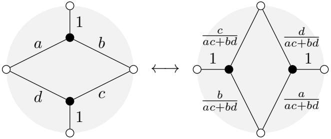


Leaf reduction

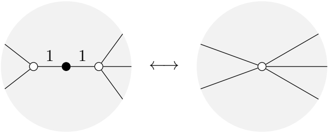
Dipole reduction

- Each reduction move preserves the boundary measurement map.
- A graph is **move-reduced** if no reduction move can be applied.

# Equivalence moves




Spider move




Contraction-uncontraction move

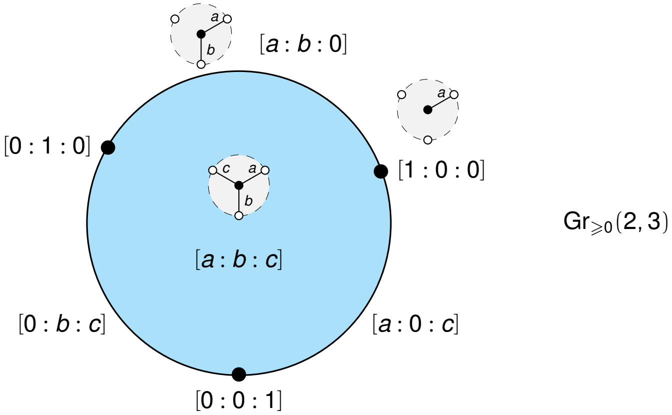
- Both moves come with transformations of weights such that the boundary measurement is preserved.
- Two move-reduced graphs are called **move-equivalent** if they are related by moves.

# Positroid stratification [Postnikov, 2006]

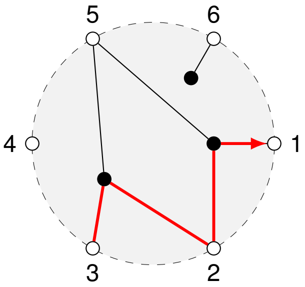
- The totally nonnegative Grassmannian has a stratification into cells called **positroid cells.**
- For move-reduced graphs,

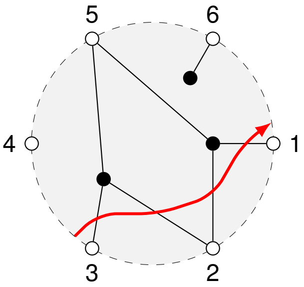
$$\text{boundary measurement} : \text{weights}/\text{gauge} \to \text{positroid cell}$$

is a homeomorphism.

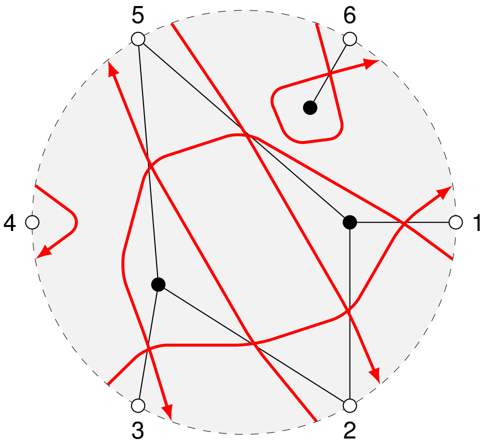


# Zig-zag paths



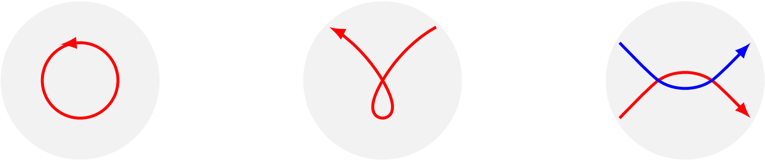
# Decorated permutation [Postnikov, 2006]




decorated permutation $= 351\overset{\circ}{4}2\overset{\bullet}{6}$

# Reduced graphs and minimal graphs

A graph is called **reduced** if the following structures are absent (except for boundary leaves).



A (leafless) graph (with no isolated components) is called **minimal** if it has the fewest number of faces among all graphs with the same decorated permutation.

**Theorem:** [Postnikov, 2006 and Thurston, 2004]

Assume the graph has a dimer cover. Then,

$$\text{move-reduced} \iff \text{reduced} \iff \text{minimal}.$$

# Classification of positroid cells

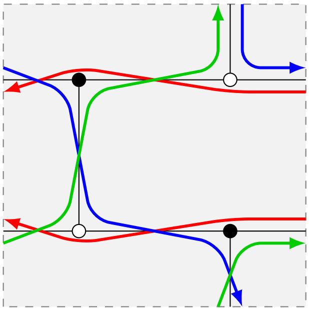
**Theorem:** [Postnikov, 2006] The following objects are in bijection:

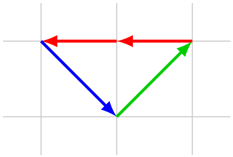
1. Positroid cells in the totally nonnegative Grassmannian.
2. Move-equivalence classes of move-reduced graphs.
3. Decorated permutations.
4. Many more...

**Question:** How to generalize these results to the dimer model on the torus.

A starting point is to classify move-equivalence classes of move-reduced graphs on the torus.

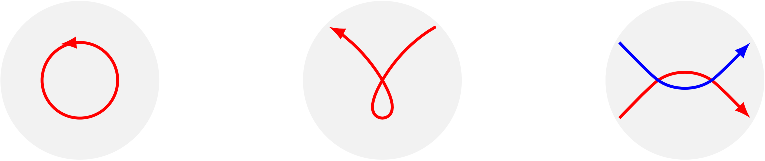
# Newton polygon (analogous to decorated permutation)





# Reduced graphs on the torus

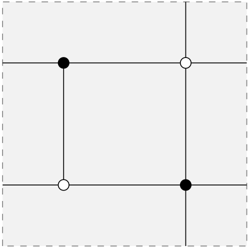
A graph on the torus is called **reduced** if in the corresponding biperiodic graph in the plane, the following structures are absent.

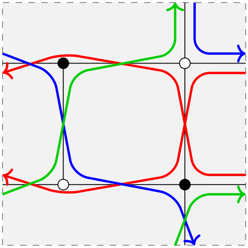


**Theorem:** [Goncharov–Kenyon, 2013]

Move-equivalence classes of reduced graphs are in bijection with convex polygons with integer vertices.

# Move-reduced $\not\Rightarrow$ Reduced

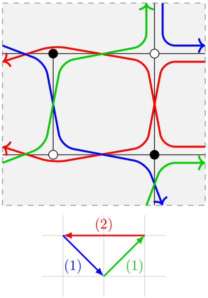


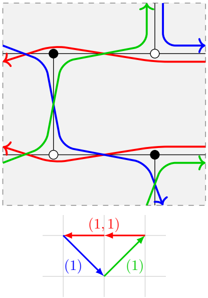


We need to generalize the Newton polygon.

# Decorated polygons

[Galashin–G, 2022] In a move-reduced graph, parallel zig-zag paths do not intersect $\implies$ Parallel zig-zag paths have a well-defined cyclic order.




Newton polygon + collection of cyclic compositions

cyclic composition = composition modulo cyclic shift (so (2,1,3,1) = (1,3,1,2), etc)

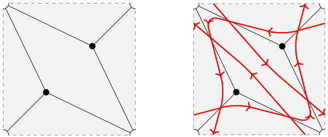
# Minimal graphs

A (leafless) graph (with no contractible components) is called **minimal** if it has the fewest number of faces among all graphs with the same decorated polygon.

**Theorem 1:** [Galashin–G, 2022] Assume the graph has a dimer cover. Then,

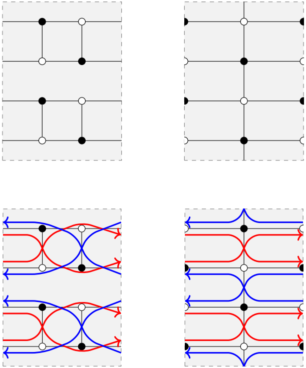
$$\text{reduced} \implies \text{move-reduced} \iff \text{minimal}.$$

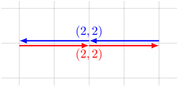
The assumption that the graph has a dimer cover is essential. The following graph is move-reduced but not minimal.



**Open problem:** Is there a generalization of reduced that is equivalent to move-reduced?

# However...





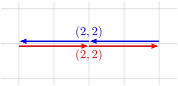

Both graphs have the same decorated Newton polygon, but are not move-equivalent since one of them is connected and the other one is not.

# Main theorem

For a cyclic composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of $n$, let $\mathrm{rot}(\alpha)$ be the smallest cyclic rotation that fixes $\alpha$.

- $\mathrm{rot}(1, 1, 1, 1, 1, 1) = 1$.
- $\mathrm{rot}(2, 1, 2, 1) = 3$.
- $\mathrm{rot}(2, 2, 1, 1) = 6$.

Let $d$ denote the gcd of rot($\alpha$) as $\alpha$ varies over the cylic compositions associated to edges of the Newton polygon. For example, $d = 2$ for the following decorated polygon:




**Theorem 2:** [Galashin–G, 2022] For any decorated polygon, there are $d$ move-equivalence classes of move-reduced graphs classified by their **modular invariant**.

# Analogies

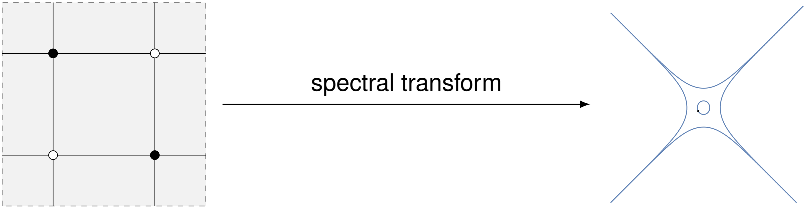
| disk | torus |
|---|---|
| move-reduced | move-reduced |
| minimal | minimal |
| reduced | ? |
| decorated permutation | decorated polygon $+$ modular invariant |
| boundary measurement | ? |
| positroid cell | ? |
| TNN Grassmannian | ? |

# Spectral transform

- The point in $\mathrm{Gr}_{\geqslant 0}(k, n)$ given by boundary measurement is the kernel of the Kasteleyn matrix $K$ (= space of discrete holomorphic functions).
- On the torus, the Kasteleyn matrix $K$ is a matrix of Laurent polynomials $K(z, w)$.
- The map

$$\text{spectral transform} : \text{weights}/\text{gauge} \mapsto \text{kernel of } K(z, w)$$

was defined by [Kenyon–Okounkov, 2007].




Harnack curve + standard divisor

# Spectral transform

**Theorem:** [Postnikov, 2006]

For move-reduced graphs,

$$\text{boundary measurement} : \text{weights}/\text{gauge} \to \text{positroid cell}$$

is a homeomorphism.

**Theorem:** [Kenyon–Okounkov, 2007]

For reduced graphs,

$$\text{spectral transform} : \text{weights}/\text{gauge} \to \text{Harnack curves with given Newton polygon} +$$

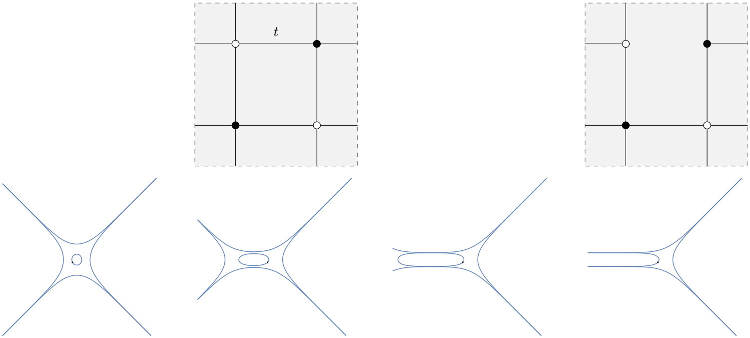
is a homeomorphism.

The explicit inverse appears in [Fock, 2015] for complex weights and [Boutillier–Cimasoni–de Tilière, 2023] for positive weights.

# Analogies

| disk | torus |
|---|---|
| move-reduced | move-reduced |
| minimal | minimal |
| reduced | ? |
| decorated permutations | decorated polygons + modular invariant |
| boundary measurement | spectral transform |
| positroid cell | space of Harnack curves + standard divisors (for reduce |
| TNN Grassmannian | compactification of space of Harnack curves + standard |

# Spectral transform for move-reduced graphs?




$t \to 0$

THANK YOU!