Delocalisation of height functions

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joint work with Piet Lammers

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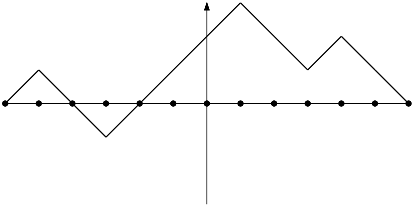
– Dimers ANR final conference (Paris) –

Delocalisation

1D time:
Random Walk → Brownian bridge
$\mathrm{Var}_n(h(0)) \sim n$.



2D time: graph homomorphisms $\mathbb{Z}^2 \to \mathbb{Z}$

$$h(u) - h(v) = \pm 1, \text{ if } u \sim v.$$

- $\mathbb{P}(h)$ =uniform;
- favour *flat points*: $\mathbb{P}(h) \propto c^{\#\text{saddle}}$;
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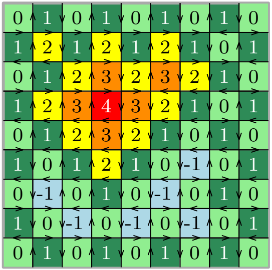

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Localisation: $\forall n \quad \mathrm{Var}_n(h(0)) \leq C$.

$\geq$**3D time**: localisation expected. **[Peled '10]**: uniform measure, $d \gg 3$.

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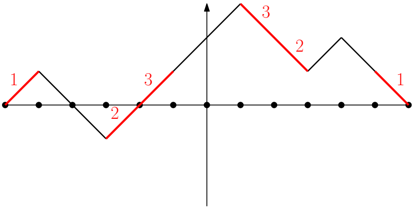

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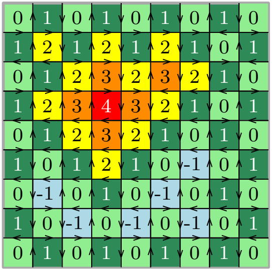

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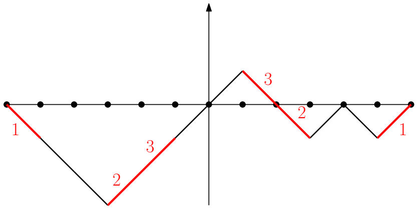

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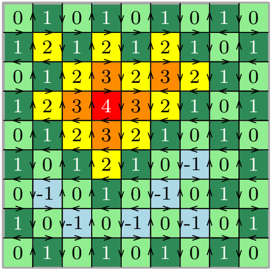

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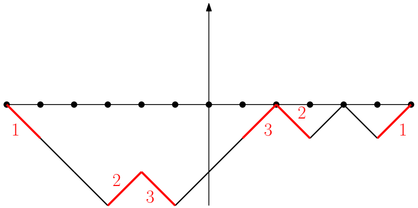

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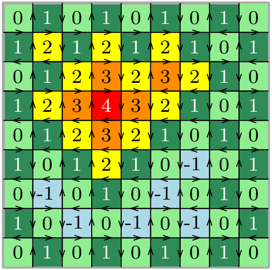

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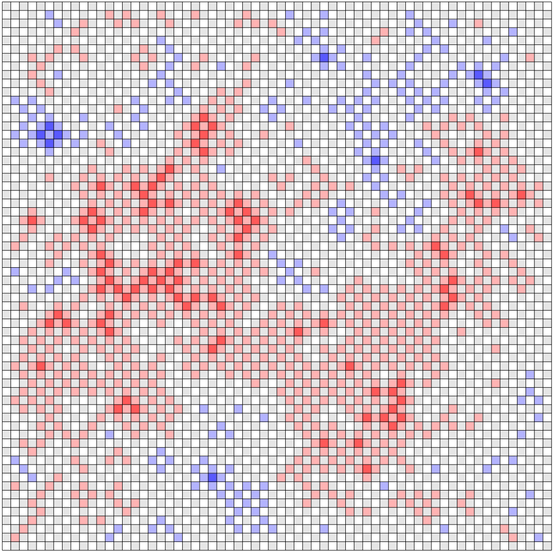

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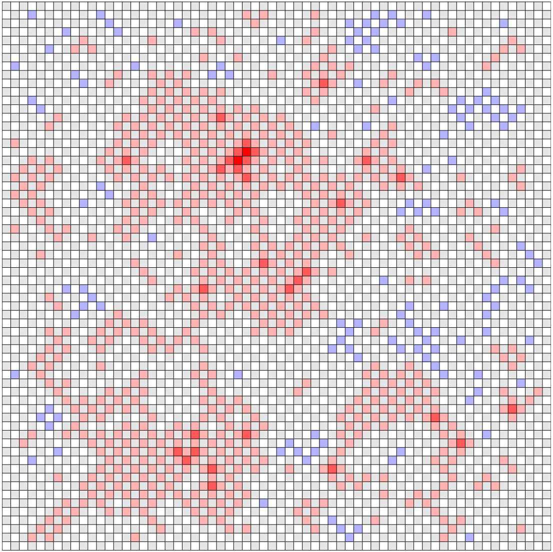

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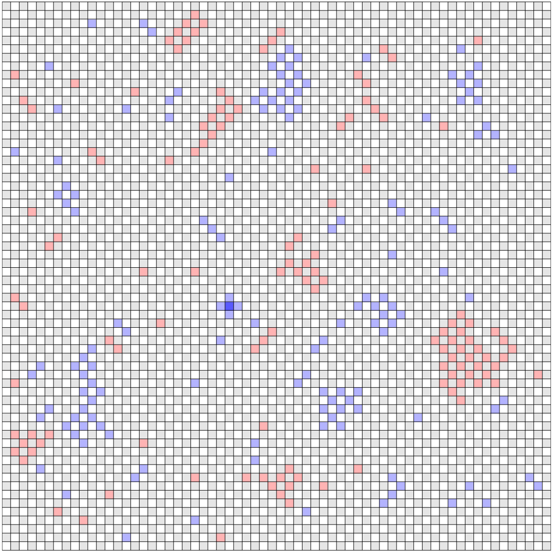

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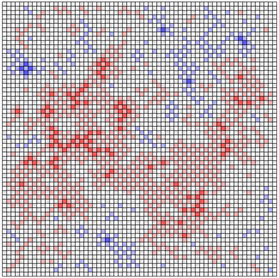

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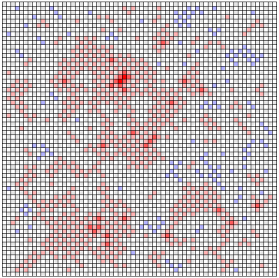

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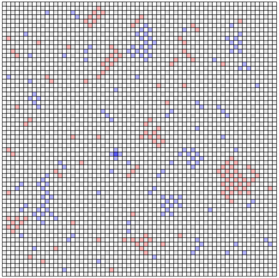

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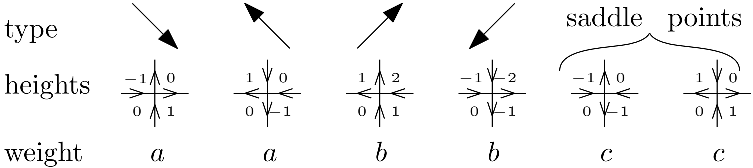
Six-vertex model, $a, b \leq c$

Gradient field:



Ice rule: two incoming + two outgoing edges.
Six local edge orientations.

$$\mathbb{P}(h) \propto a^{\#\{\searrow,\nwarrow\}} \cdot b^{\#\{\nearrow,\swarrow\}} \cdot c^{\#\text{saddle}}.$$

Prop (positive association: FKG inequality)

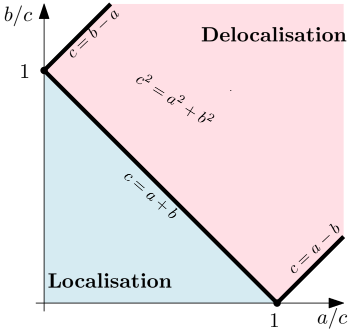
Let $a, b \leq c$. Then, for any increasing F, G,

$$\mathbb{E}(F(h) \cdot G(h)) \geq \mathbb{E}(F(h)) \cdot \mathbb{E}(G(h)).$$

[Fortuin–Kasteleyn–Ginibre'72], [Benjamini–Haggström–Mossel'00]

Background

- free energy computation
  [Yang-Yang '66], [Sutherland '67], [Lieb '67]



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- dimers $\to$ GFF: $a^2 + b^2 = c^2$
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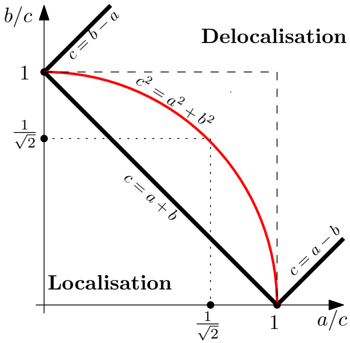

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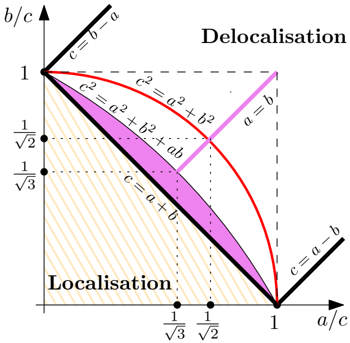
- Localisation: $a + b < c$
  [Duminil-Copin–Gagnebin–Harel–Manolescu–Tassion'16], [Ray–Spinka'19],[G.–Peled'19]
- log-Delocalisation: $a = b \leq c \leq a + b$
  [Lis'20],
  [Duminil-Copin–Karrila–Manolescu–Oulamara'20]
- Rotational invariance:
  $\sqrt{a^2 + b^2 + ab} \leq c \leq a + b$
  [Duminil-Copin–Kozlowski–Krachun–Manolescu–Oulamara'20]

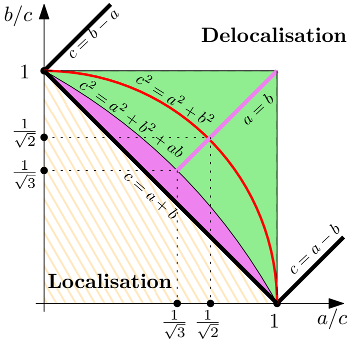


Result and main ideas

Theorem (G.–Lammers '23)

Delocalisation for all $a, b \leq c \leq a + b$. If $a = b$: log-*delocalisation.*

Still open: log-bound when $a \neq b$.



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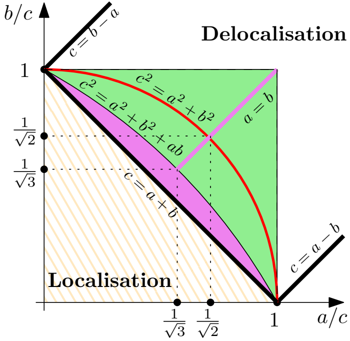
Main ideas:

- spin representation $\mathrm{mod}\ 4$ (aka Ashkin–Teller) **[Rys'63]**;
- two representations + duality **[Lis'19]**;
- joint FKG inequality: spins + edges (aka **[Lammers–Ott'21]**);
- ergodicity, non-coexistence theorem **[Zhang'90s]**, **[Sheffield'05]**;
- $\mathbb{T}$-circuit argument **[G.–Peled'19]**: no half-ordered measures
- log-bound: dichotomy **[G.–Manolescu'18]**.



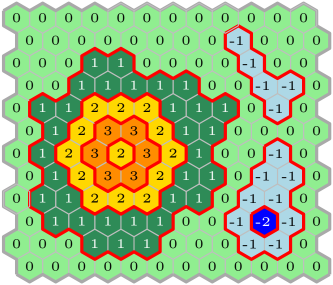
Lipschitz functions; loop $O(n)$ model at $n = 2$

$h \colon \mathrm{Faces}(\mathrm{Hex}) \to \mathbb{Z}$, so that $h(u) - h(v) \in \{0, \pm 1\}$ if $u \sim v$. Measure:

$$\mathbb{P}(h) \propto x^{\#\{u \sim v \colon h(u) \neq h(v)\}} \quad \Rightarrow \quad \mathbb{P}(\text{level lines}) \propto 2^{\#\text{loops}} x^{\#\text{edges}}$$

[Duminil-Copin–G.–Peled–Spinka'17], [G., Manolescu'18]:

Localisation for $0 < x < 1/\sqrt{3} + \varepsilon$. log-**Delocalisation** at $x = 1/\sqrt{2}$ and $x = 1$.



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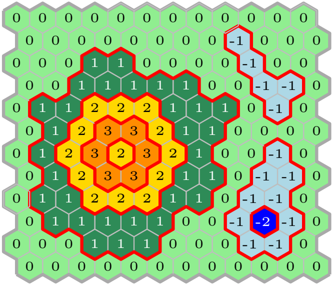

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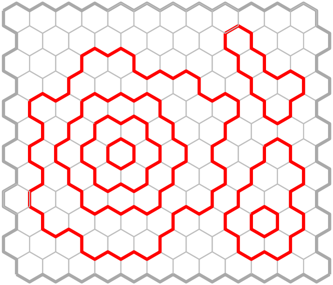

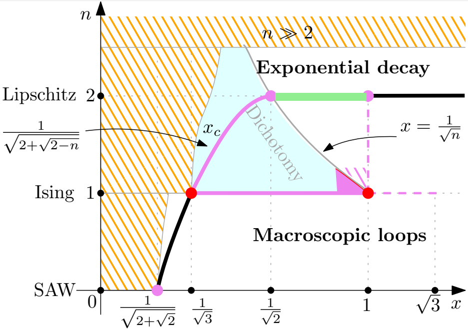

Random-cluster model: continuity of the phase transition

Rectangular lattice $\mathbb{L}$, $p_v, p_h \in (0,1)$, $q > 0$. Box $\Lambda_n = (V,E) \subset \mathbb{L}$.
For a percolation configuration $\omega \in \{\text{closed}, \text{open}\}^E$,

$$\mathbb{P}_n(\omega) \propto p_v^{\#\text{open}_v} \cdot (1-p_v)^{\#\text{closed}_v} \cdot p_h^{\#\text{open}_h} \cdot (1-p_h)^{\#\text{closed}_h} \cdot q^{\#\text{clusters}}.$$

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$q \geq 1$: **FKG** inequality $\Rightarrow$ the weak limit $\mathbb{P}_n \to \mathbb{P}^{\text{free}}$ is well-defined.
Also the wired measure: $\mathbb{P}_n(\cdot \,|\, \omega|_{\partial\Lambda_n} \equiv \text{open}) \to \mathbb{P}^{\text{wired}}$.
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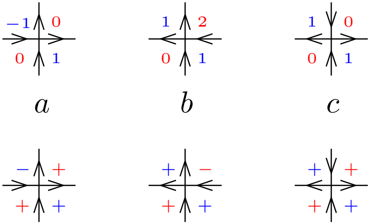
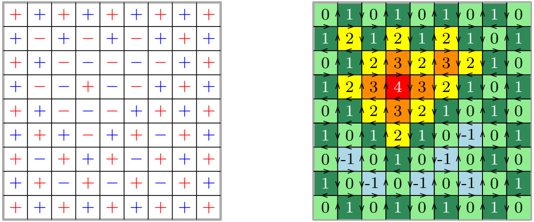
Theorem (G.–Lammers '23)

Let $1 \leq q \leq 4$. Then, $\mathbb{P}^{wired} = \mathbb{P}^{free}$. No infinite cluster at the self-dual line.

Not a new result:
[Duminil-Copin–Sidoravicius–Tassion'15], [Duminil-Copin–Li–Manolescu'17]
New proof:
no use of parafermionic observable, Bethe Ansatz, Yang–Baxter.

Coupled $\mathrm{ES}^{\text{even}}$ and ES^{odd} edge configurations **[Lis'19]**.





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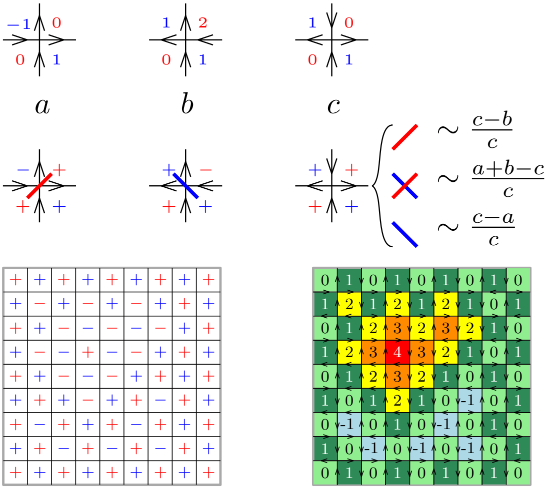

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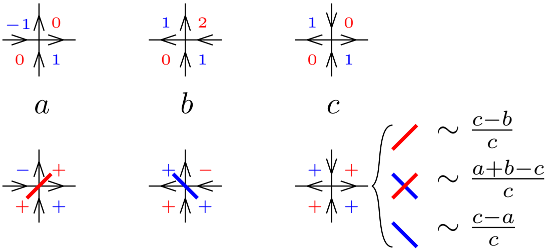

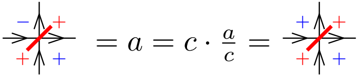

even edges decouple

odd spins:

circuits are

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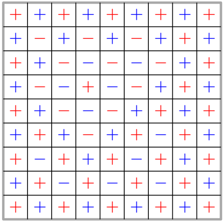

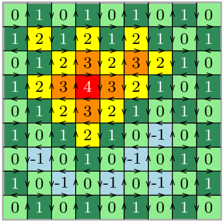

Super-duality: $(\mathrm{ES}^{\text{even}})^* \subseteq \mathrm{ES}^{\text{odd}}$ if $c \leq a + b$.

Goal: find ∞ many circuits of $\mathrm{ES}^{\text{even}}$ and ES^{odd}.

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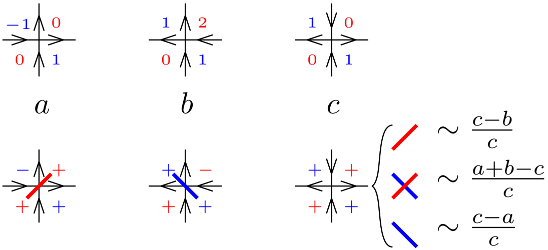
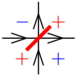
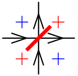


 $= a = c \cdot \frac{a}{c} =$ 

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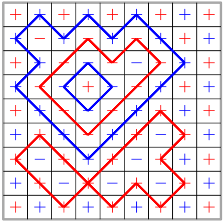

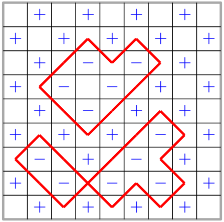

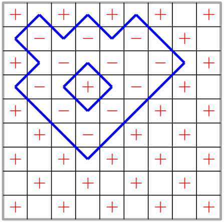

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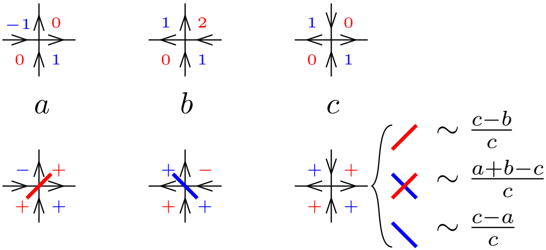

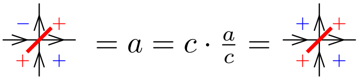

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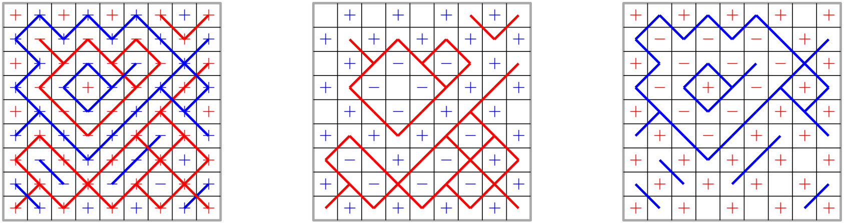

Super-duality: $(\mathrm{ES}^{\text{even}})^* \subseteq \mathrm{ES}^{\text{odd}}$ if $c \leq a + b$.

Goal: find ∞ many circuits of $\mathrm{ES}^{\text{even}}$ and ES^{odd}.

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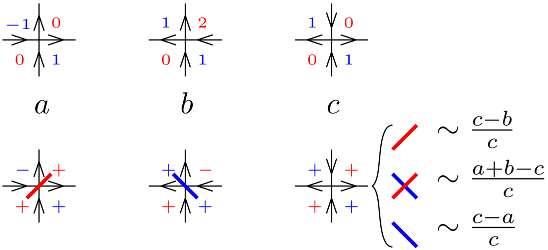

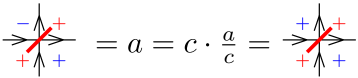

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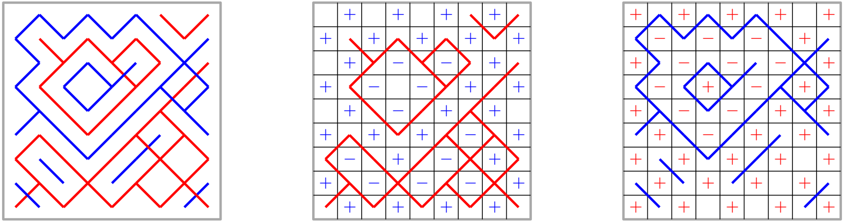

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Step 2: joint FKG property

Spin config.: σ^{even} and σ^{odd}. Edge config.: $\mathrm{ES}^{\text{even}}$ and ES^{odd}.
Note: $\sigma^{\text{even}} \equiv$ const on clusters of $\mathrm{ES}^{\text{even}}$. Define: $\mathrm{ES}^{\text{even}+} \sqcup \mathrm{ES}^{\text{even}-} = \mathrm{ES}^{\text{even}}$.

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Prop (G.–Lammers '23)

Let $a, b \leq c$. The triple $(\sigma^{\text{even}}, \mathrm{ES}^{\text{even}+}, -\mathrm{ES}^{\text{even}-})$ satisfies the FKG inequality:

$$\mathbb{E}[F(\sigma^{\text{even}}, \mathrm{ES}^{\text{even}}) \cdot G(\sigma^{\text{even}}, \mathrm{ES}^{\text{even}})] \geq \mathbb{E}[F(\sigma^{\text{even}}, \mathrm{ES}^{\text{even}})] \cdot \mathbb{E}[G(\sigma^{\text{even}}, \mathrm{ES}^{\text{even}})],$$

for any F, G increasing in σ^{even} and $\mathrm{ES}^{\text{even}+}$ and decreasing in $\mathrm{ES}^{\text{even}-}$.

[Lis'19], [Ray–Spinka'19], [G.–Peled'19]: same for σ^{even} only.

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Proof.

1. FKG for $(\mathrm{ES}^{\text{even}+}, -\mathrm{ES}^{\text{even}-})$ is satisfied by $\mathbb{P}^{\sigma} := \mathbb{P}(\cdot \mid \sigma^{\text{even}} = \sigma)$;
2. the law of $(\mathrm{ES}^{\text{even}+}, -\mathrm{ES}^{\text{even}-})$ under $\mathbb{P}^{\sigma}$ is $\nearrow$ in σ;
3. $\mathbb{E}[F \cdot G] = \int \mathbb{P}^{\sigma}(F \cdot G) \geq \int \mathbb{P}^{\sigma}(F) \cdot \mathbb{P}^{\sigma}(G) \geq \int \mathbb{P}^{\sigma}(F) \cdot \int \mathbb{P}^{\sigma}(G) = \mathbb{E}[F] \cdot \mathbb{E}[G]$.

□

Step 3: Markov property and $(\sigma^{\text{even}}, \mathrm{ES}^{\text{even}})$-ergodicity

Circuits of $\mathrm{ES}^{\text{even}+}$ are **domain Markov**: for $\Omega \subset \Lambda$,

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$\Rightarrow$ **monotonicity** in Ω: $\mu^+_\Omega \searrow$ (stochastically) as $\Omega \nearrow \mathbb{Z}^2$.
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2. define $\mathbb{P}^+$ given μ^+: same as above;
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NB: so far no ergodicity in $(\sigma^{\text{odd}}, \mathrm{ES}^{\text{odd}})$!
Need to rule out an infinite cluster in $(\mathrm{ES}^{\text{even}})^*$.

Step 4: Non-coexistence + super-duality $\Rightarrow$ full ergodicity

Theorem (Zhang'90s; Sheffield '05)

If μ is a probability measure on $\{0,1\}^{E(\mathbb{Z}^2)}$ that is FKG and shift invariant, then

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However, by the **non-coexistence**, $\mathbb{P}^+(\mathrm{ES}^{\text{even}} \text{ has an } \infty \text{ cluster}) = 0$.
This contradicts the red/blue symmetry. Hence, $\mathbb{P}^+$ is **ergodic**.

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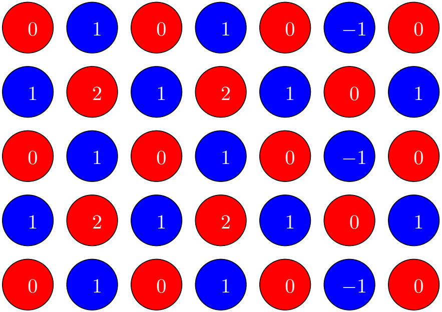

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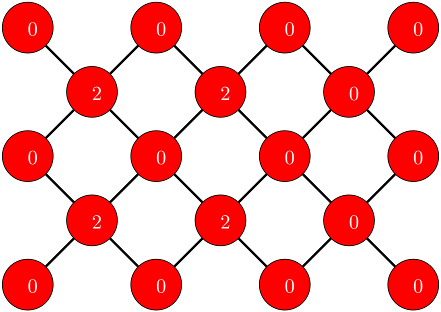

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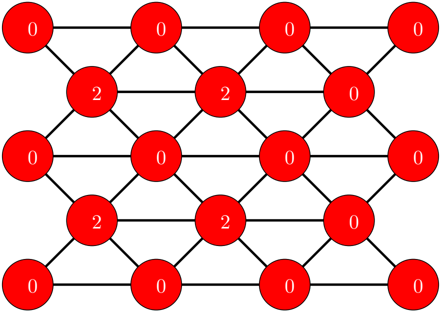

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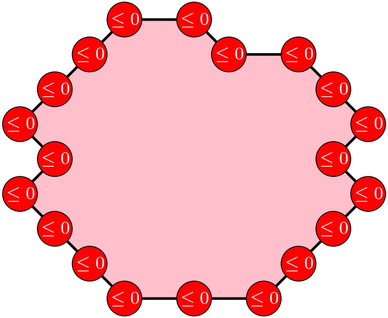

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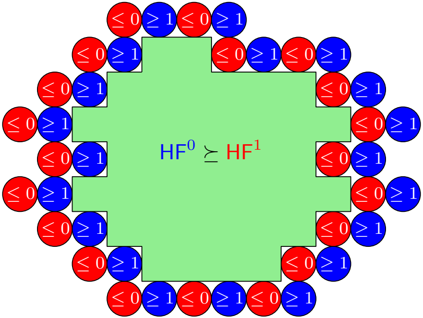


Sample $h^1 \sim \mathsf{HF}^1$.
Find the exterior-most $\mathbb{T}_{\leq 0}$-circuit.
Outside define:

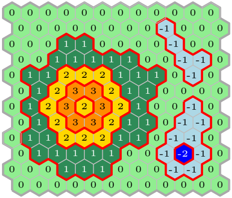
$$h^0(i,j) := 1 - h^1(i-1,j) \sim \mathsf{HF}^1.$$

Conditioned on the exterior of the circuit:

$$\mathsf{HF}^0 \succeq \mathsf{HF}^1 \quad \text{in the interior.}$$

Lipschitz functions: site percolation duality

h mod 4	0	1	2	3
red spin	+	+	−	−
blue spin	+	−	−	+



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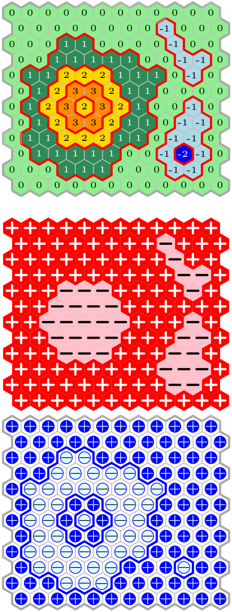

Lipschitz functions: site percolation duality

Δ-triangles in $(\mathrm{Hex})^*$.
They form a triangular lattice.

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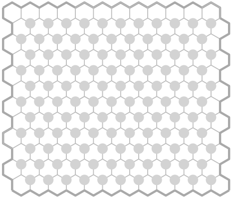

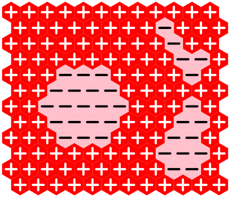

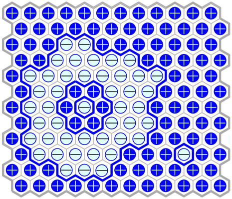

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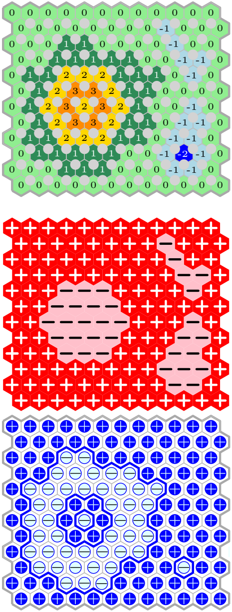

$$\mathbb{P}(h) \propto x^{\#\{u\sim v:\ h_u\neq h_v\}} = (x^2)^{\#\{uvw\in\Delta:\ h|_{uvw}\neq\mathrm{const}\}}$$

$$= (x^2)^{|\Delta\cap\{\text{red loops}\}|}\cdot(x^2)^{|\Delta\cap\{\text{blue loops}\}|}.$$

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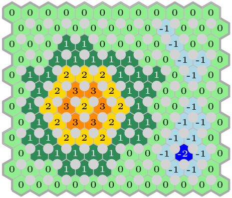

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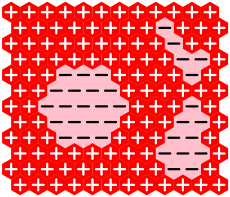
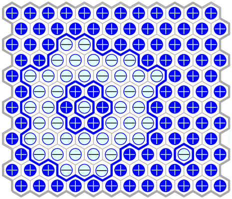
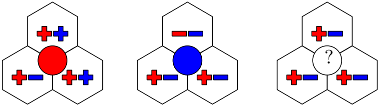
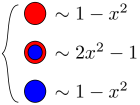






x^2 $\quad$ x^2 $\quad$ 1



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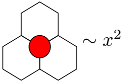

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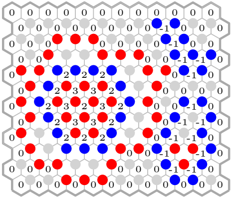

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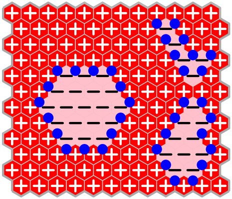
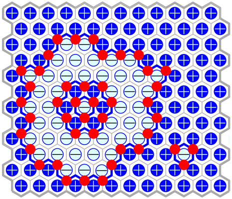
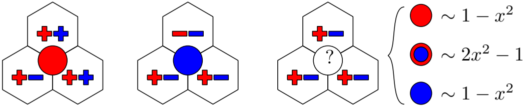
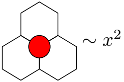




x^2 x^2 1



decoupling:



Lipschitz functions: site percolation duality

Δ-triangles in (Hex)*.
They form a triangular lattice.
Rewrite $\mathbb{P}(h)$:

h mod 4	0	1	2	3
red spin	+	+	−	−
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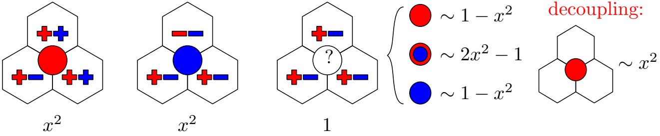

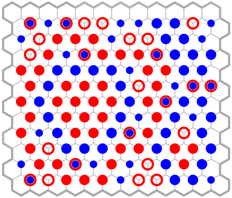

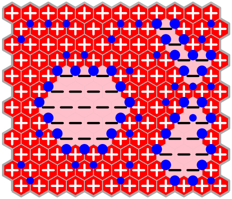

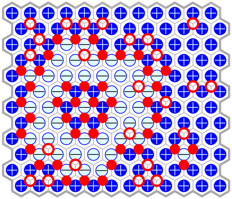

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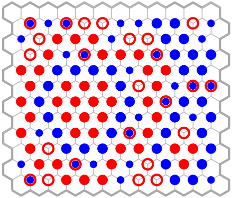

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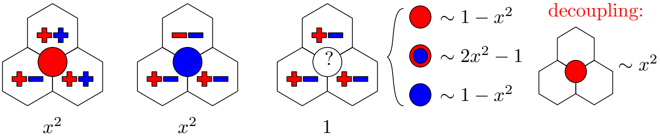

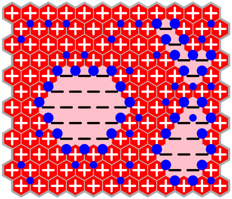

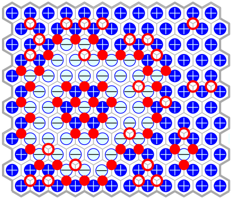

FKG for $(\sigma, \mathrm{ES}^+, -\mathrm{ES}^-)$. **Super-duality** when $x^2 \geq 1/2$.
As before: the limit $\mathbb{P}^+_n \to \mathbb{P}^+$ is ergodic.
Non-coexistence: $\{\sigma = +\}$ and $\{\sigma = -\}$ don't percolate.
$\Rightarrow$ there are ∞ many blue loops $\Rightarrow$ same for red loops.

Random-cluster model: BKW correspondence

[Temperley–Lieb'71], [Baxter–Kelland–Wu'76]

Symmetric: $a = b = 1$, $p = p_{\mathrm{sd}} = \frac{\sqrt{q}}{\sqrt{q}+1}$. Write $\sqrt{q} = 2\cos\lambda$.

$$\mathbb{P}(\omega) \propto p^{\#\text{open}}(1-p)^{\#\text{closed}} q^{\#\text{clusters}} \propto \sqrt{q}^{\#\text{loops}} = \sum_{\vec{\eta}\perp\omega} e^{i\lambda(\circlearrowleft - \circlearrowright)}$$

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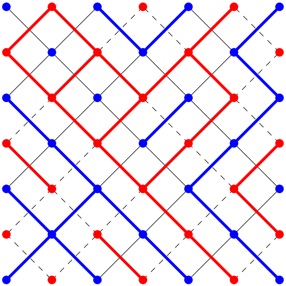

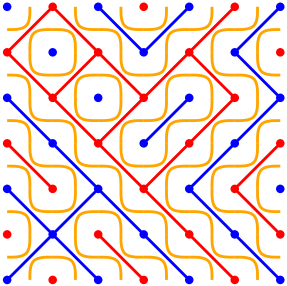

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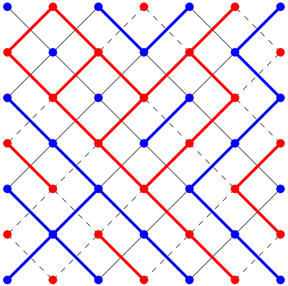

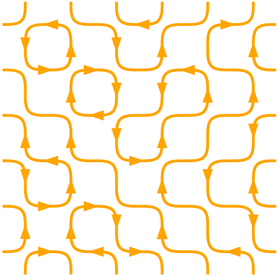

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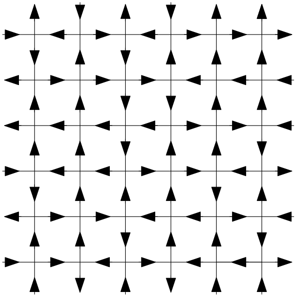

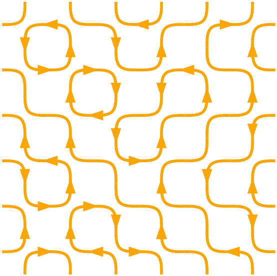

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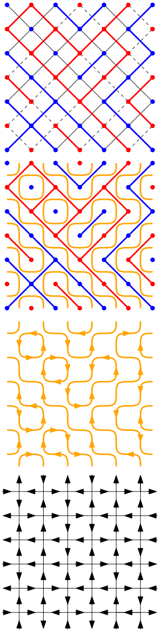
Same holds with a defect line **[Dubédat'11]**:

$$\mathbb{E}_{6\mathrm{V}}[e^{i\alpha(h(u)-h(v))}] = \mathbb{E}_{\mathrm{RCM}}[F_{\lambda,\alpha}(\#\mathrm{loops}(u), \#\mathrm{loops}(v))],$$

where $F_{\lambda,\alpha}(x,y) = \cos^x(\lambda+\alpha)\cdot\cos^y(\lambda-\alpha)/\cos^{x+y}\lambda$.



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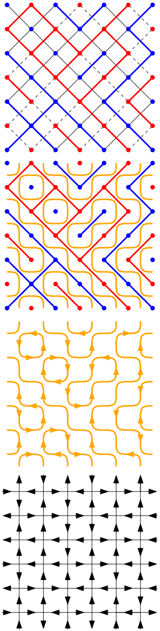

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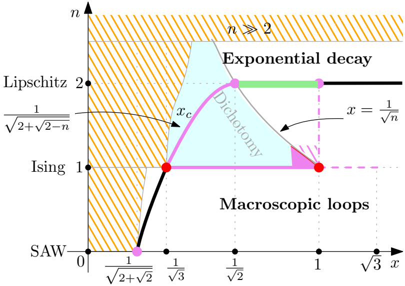
Note: $\lambda \in [0, \pi/3]$. Fix $\alpha = \pi/8 \in (0, \pi/6)$. Then,

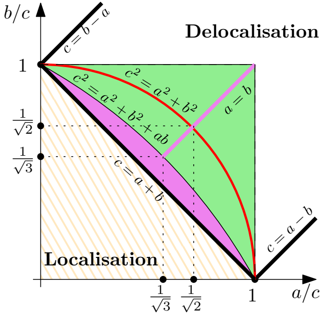
$$\mathbb{E}_{6\mathrm{V}}[e^{i\alpha(h(u)-h(v))}] \geq \mathbb{P}_{\mathrm{RCM}}(u \leftrightarrow v).$$

By **delocalisation**: $\mathbb{E}_{6\mathrm{V}}[e^{i\alpha(h(u)-h(v))}] \to 0$, as $|u-v| \to \infty$.

Summary:

1. six-vertex and Lipschitz together;
2. RCM: continuity without integrability;
3. joint FKG: spins + Edwards–Sokal;
4. non-coexistence theorem;
5. $\mathbb{T}$-circuit argument;
6. Fourier transform of heights $\leftrightarrow$ loops.





Discussion

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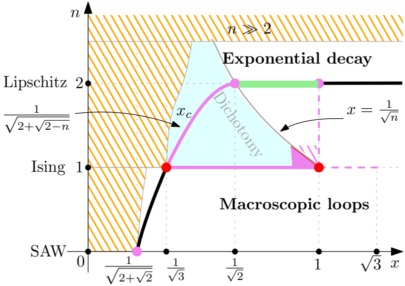

Future directions:

1. Lipschitz: localisation when $x < 1/\sqrt{2}$?
2. Lipschitz functions on $\mathbb{Z}^2$?
3. Loop $O(n)$ at $x_c(n)$ without integrability?
4. RSW/dichotomy without $pi/2$ rotations: log-deloc when $a \neq b$?
5. random-cluster model: $q < 1$?