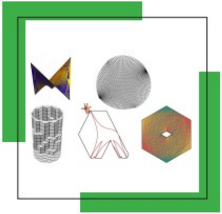
ipam institute for pure & applied mathematics

Geometry, Statistical Mechanics, and Integrability

March 11 - June 14, 2024



Long Program Schedule

- Opening Day: March 11, 2024
- Geometry, Statistical Mechanics, and Integrability Tutorials: March 12-15, 2024
- Workshop I: Statistical Mechanics and Discrete Geometry: March 25-29, 2024
- Workshop II: Integrability and Algebraic Combinatorics: April 15-19, 2024
- Workshop III: Statistical Mechanics Beyond 2D: May 6-10, 2024
- Workshop IV: Vertex Models: Algebraic and Probabilistic Aspects of Universality: May 20-24, 2024
- Culminating Workshop at Lake Arrowhead: June 9-14, 2024

Organizers

Dmitry Chelkak (Uni of Michigan)
Jan de Gier (Univ. Melbourne)
Vadim Gorin (UC Berkeley)
Richard Kenyon (Yale)
Greta Panova (USC)
Sanjay Ramassamy (CNRS)
Marianna Russkikh (Caltech)

Apply before October 11, 2023 at www.ipam.ucla.edu/gsi2024

Boundary limits for the six-vertex model

Vadim Gorin

UC Berkeley

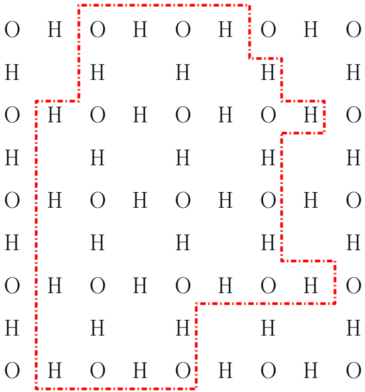
July, 2023

Six–vertex model

O	H	O	H	O	H	O	H	O
H		H		H		H		H
O	H	O	H	O	H	O	H	O
H		H		H		H		H
O	H	O	H	O	H	O	H	O
H		H		H		H		H
O	H	O	H	O	H	O	H	O
H		H		H		H		H
O	H	O	H	O	H	O	H	O

Square grid with O in the vertices and H on the edges.

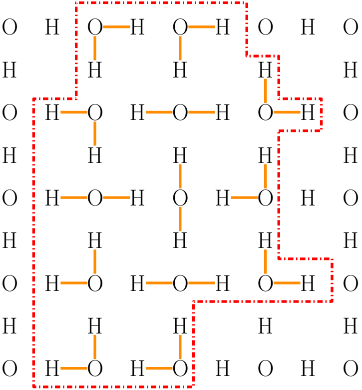
Six–vertex model



Square grid with O in the vertices and H on the edges.

Take a finite/infinite domain.

Six–vertex model

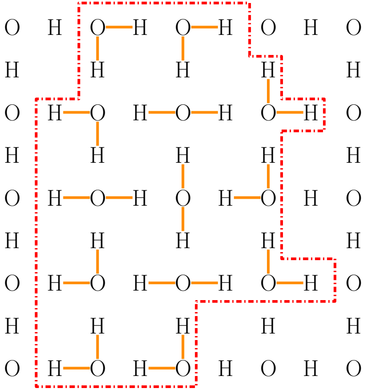


Square grid with O in the vertices and H on the edges.

Take a finite/infinite domain.

Configurations: possible matchings of *all* atoms inside domain into H_2O molecules.

Six–vertex model

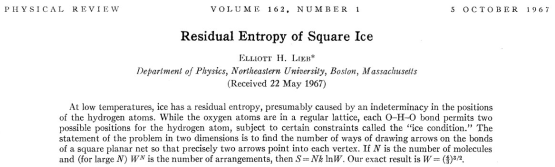


Square grid with O in the vertices and H on the edges.

Take a finite/infinite domain.

Configurations: possible matchings of *all* atoms inside domain into H_2O molecules.

This is **square ice model**. Real-world ice has somewhat similar (although 3d) structure.



PHYSICAL REVIEW VOLUME 162, NUMBER 1 5 OCTOBER 1967

Residual Entropy of Square Ice

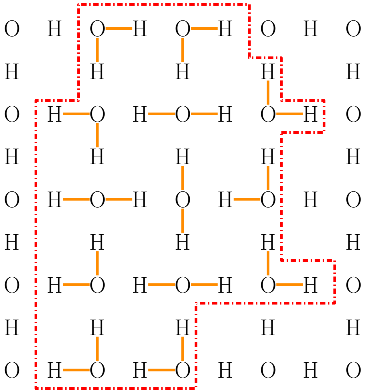
Elliott H. Lieb*

Department of Physics, Northeastern University, Boston, Massachusetts

(Received 22 May 1967)

At low temperatures, ice has a residual entropy, presumably caused by an indeterminacy in the positions of the hydrogen atoms. While the oxygen atoms are in a regular lattice, each O–H–O bond permits two possible positions for the hydrogen atom, subject to certain constraints called the "ice condition." The statement of the problem in two dimensions is to find the number of ways of drawing arrows on the bonds of a square planar net so that precisely two arrows point into each vertex. If N is the number of molecules and (for large N) W^N is the number of arrangements, then $S = Nk \ln W$. Our exact result is $W = (\frac{4}{3})^{3/2}$.

Six–vertex model



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Take a finite/infinite domain.

Configurations: possible matchings of *all* atoms inside domain into H_2O molecules.

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Published: 25 March 2015

Square ice in graphene nanocapillaries

G. Algara-Siller, O. Lehtinen, F. C. Wang, R. R. Nair, U. Kaiser, H. A. Wu, A. K. Geim & I. V. Grigorieva

Nature **519**, 443–445 (2015) | Cite this article

43k Accesses | **543** Citations | **277** Altmetric | Metrics

Published: 23 December 2015

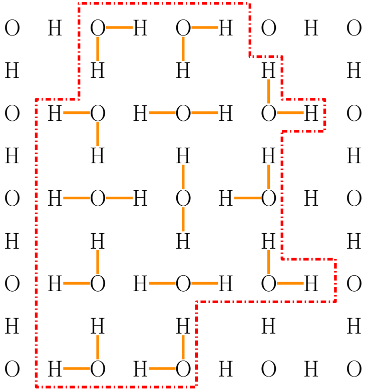
The observation of square ice in graphene questioned

Wu Zhou, Kuibo Yin, Cunhui Wang, Yuyang Zhang, Tao Xu, Albina Borisevich, Litao Sun, Juan Carlos Idrobo, Matthew F. Chisholm, Sokrates T. Pantelides, Robert F. Klie & Andrew R. Lupini

Nature **528**, E1–E2 (2015) | Cite this article

12k Accesses | **85** Citations | **25** Altmetric | Metrics

Six–vertex model



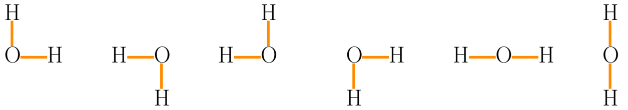
Square grid with O in the vertices and H on the edges.

Take a finite/infinite domain.

Configurations: possible matchings of *all* atoms inside domain into H_2O molecules.

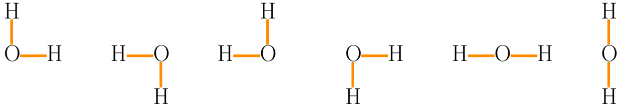
This is **square ice model**. Real-world ice has somewhat similar (although 3d) structure.

Also known as the **six vertex model**.



Gibbs measures

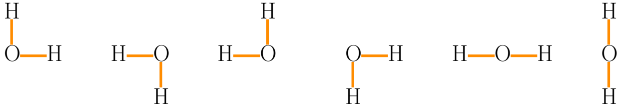
Six positive weights corresponding to types of vertices.



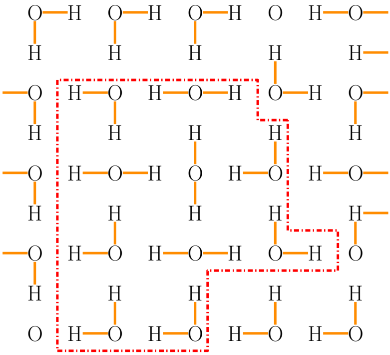
a_1 a_2 b_1 b_2 c_1 c_2

Gibbs measures

Six positive weights corresponding to types of vertices.



a_1 a_2 b_1 b_2 c_1 c_2



Gibbs probability measure on configurations:

$$\frac{a_1^{\#(a_1)} a_2^{\#(a_2)} b_1^{\#(b_1)} b_2^{\#(b_2)} c_1^{\#(c_1)} c_2^{\#(c_2)}}{Z(\Omega; a_1, a_2, b_1, b_2, c_1, c_2)}$$

Gibbs measures

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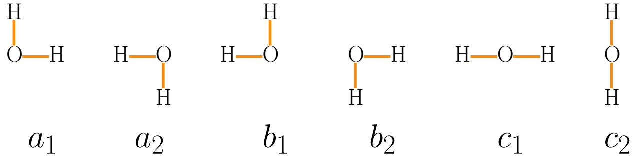

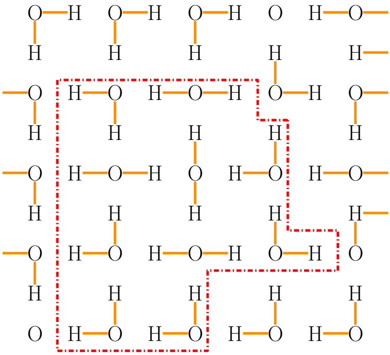

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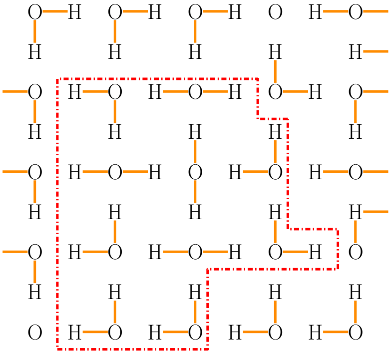
Remark. Distribution depends only on $\frac{b_1 b_2}{a_1 a_2}$ and $\frac{c_1 c_2}{a_1 a_2}$.

Gibbs measures



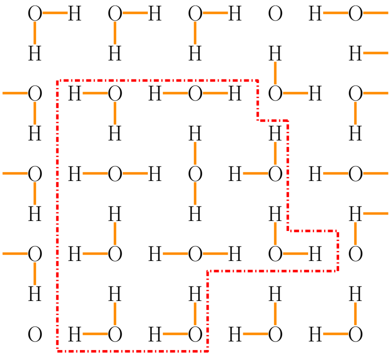
Gibbs probability measure on configurations:

$$\frac{a_1^{\#(a_1)} a_2^{\#(a_2)} b_1^{\#(b_1)} b_2^{\#(b_2)} c_1^{\#(c_1)} c_2^{\#(c_2)}}{Z(\Omega; a_1, a_2, b_1, b_2, c_1, c_2)}$$

Remark. Distribution depends only on $\frac{b_1 b_2}{a_1 a_2}$ and $\frac{c_1 c_2}{a_1 a_2}$.

Example. *Uniform* measure on configurations in a fixed domain is Gibbs with $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$.

Gibbs measures



Gibbs probability measure on configurations:

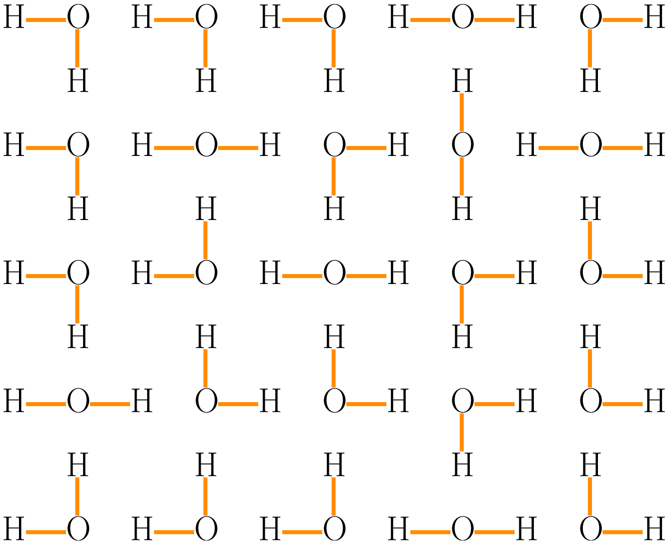
$$\frac{a_1^{\#(a_1)} a_2^{\#(a_2)} b_1^{\#(b_1)} b_2^{\#(b_2)} c_1^{\#(c_1)} c_2^{\#(c_2)}}{Z(\Omega; a_1, a_2, b_1, b_2, c_1, c_2)}$$

Remark. Distribution depends only on $\frac{b_1 b_2}{a_1 a_2}$ and $\frac{c_1 c_2}{a_1 a_2}$.

Example. *Uniform* measure on configurations in a fixed domain is Gibbs with $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$.

We aim to study asymptotic properties of Gibbs measures.

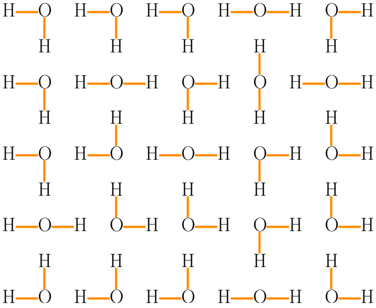
Domain wall boundary conditions (DWBC)



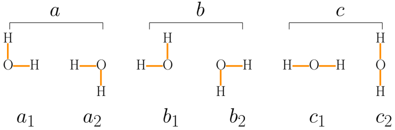
Simplest possible domain: $N \times N$ square.

Our setup: (a, b, c)–measure with DWBC.

$N \times N$ square



Symmetric weights:

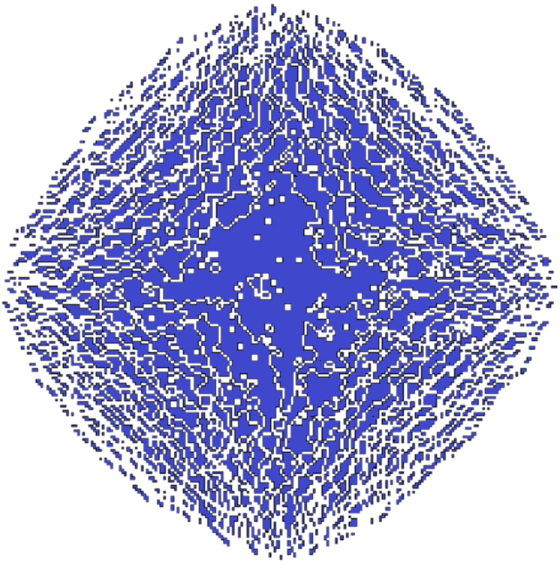


No loss of generality, because of dependence on $\frac{b_1 b_2}{a_1 a_2}$ and $\frac{c_1 c_2}{a_1 a_2}$.

How does a **random** configuration look like as $N \to \infty$?

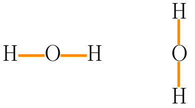
$\Delta = \frac{a^2+b^2-c^2}{2ab}$ will play a role.

$N = 200$ simulation by David Keating



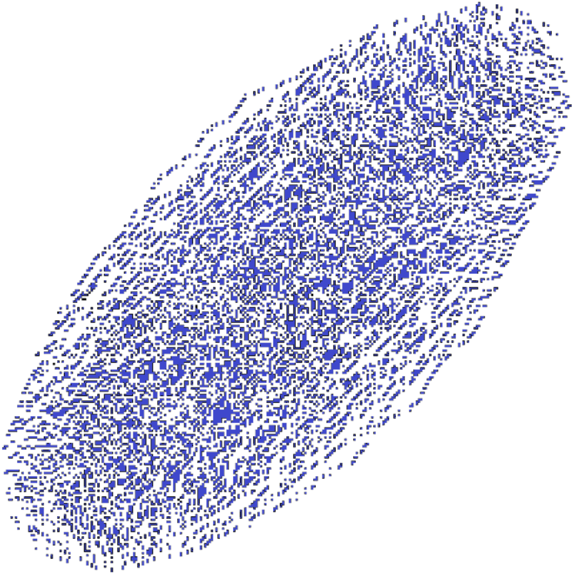
$a = 1$
$b = 1$
$c = \sqrt{8}$
$\Delta = -3$

only c-vertices shown



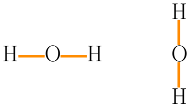
Almost **nothing** in this picture was explained rigorously.

$N = 256$ simulation by David Keating

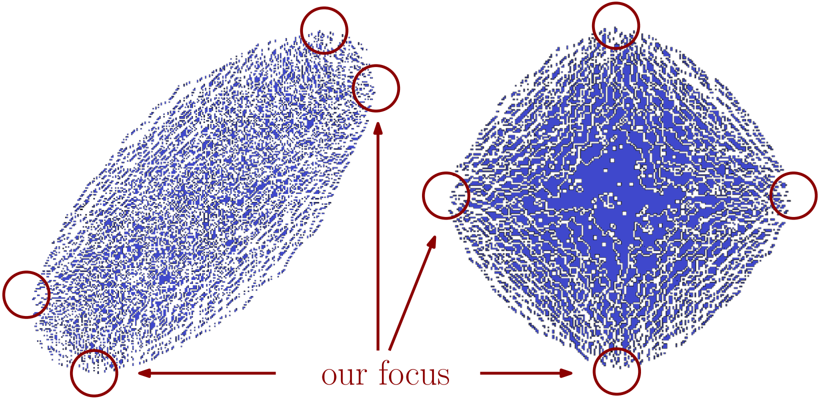


$a = 2$
$b = 1$
$c = 2$
$\Delta = \frac{1}{4}$

only c-vertices shown



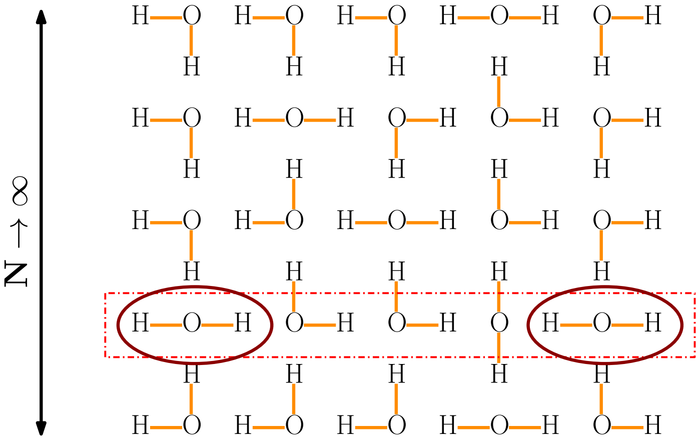
Boundary limits?



- What happens near boundaries as $N \to \infty$?
- Boundary conditions are seen **only** through these points.
- By symmetries, it is sufficient to deal with lower boundary.

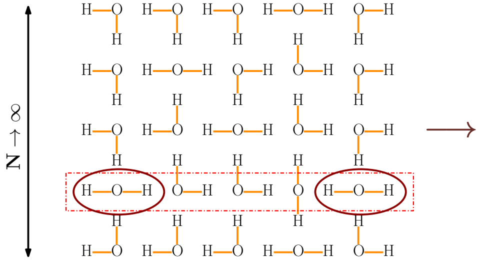
GUE for all $\Delta < 1$

Theorem. (Gorin–Liechty-23) For $\Delta < 1$ the probability that there are precisely k horizontal molecules in line k tends to 1 as $N \to \infty$.



GUE for all $\Delta < 1$

Theorem. (Gorin–Liechty-23) For $\Delta < 1$, the positions of horizontal molecules in line k, after subtracting $\mathfrak{m}(a,b,c)N$ and dividing by $\mathfrak{s}(a,b,c)\sqrt{N}$, converge in distribution to the eigenvalues of $k \times k$ matrix of **Gaussian Unitary Ensemble**.



Eigenvalues of $\frac{X+X^*}{2}$.

$X = k \times k$ matrix with i.i.d. $\mathcal{N}(0,1) + \mathbf{i}\mathcal{N}(0,1)$ elements.

- Horizontal molecules uniquely fix all others.
- **Corollary:** The first k rows $\to$ **GUE–corners process.**

GUE for all $\Delta < 1$

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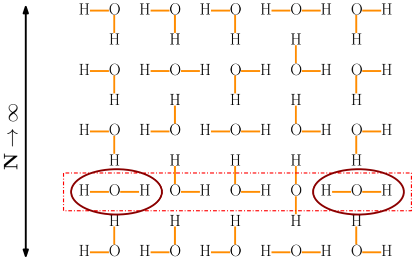

$\longrightarrow$ Eigenvalues of $\frac{X+X^*}{2}$.

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- Horizontal molecules uniquely fix all others.
- **Corollary:** The first k rows $\to$ **GUE–corners process.**
- Previous results:
  1. $\Delta = 0$: [Johansson-Nordenstam-06] through **domino tilings**.
  2. $a = b = c = 1$: [Gorin-Panova-15] through **Schur functions**.

GUE for all $\Delta < 1$

Theorem. (Gorin–Liechty-23) For $\Delta < 1$, the positions of horizontal molecules in line k, after subtracting $\mathfrak{m}(a,b,c)N$ and dividing by $\mathfrak{s}(a,b,c)\sqrt{N}$, converge in distribution to the eigenvalues of $k \times k$ matrix of **Gaussian Unitary Ensemble**.

$|\Delta| < 1$: $a = \sin(\gamma - t)$, $b = \sin(\gamma + t)$, $c = \sin(2\gamma)$, $|t| < \gamma < \pi/2$

$$\mathfrak{m}(a,b,c) = \frac{\cot(\gamma+t) + \frac{\pi}{2\gamma}\tan\left(\frac{\pi t}{2\gamma}\right)}{\cot(\gamma-t)+\cot(\gamma+t)}, \qquad \mathfrak{s}(a,b,c) = \frac{\sin(\gamma-t)\sin(\gamma+t)}{\sin(2\gamma)} \times$$

$$\times\sqrt{\frac{2}{3}\left(\frac{\pi^2}{4\gamma^2}-1\right) - \left(\cot(\gamma-t) - \frac{\pi}{2\gamma}\tan\left(\frac{\pi t}{2\gamma}\right)\right)\left(\cot(\gamma+t)+\frac{\pi}{2\gamma}\tan\left(\frac{\pi t}{2\gamma}\right)\right)}.$$

GUE for all $\Delta < 1$

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$\Delta < -1$: $a = \sinh(\gamma - t)$, $b = \sinh(\gamma + t)$, $c = \sinh(2\gamma)$, $|t| < \gamma$

$$\mathfrak{m}(a,b,c) = \frac{\coth(\gamma+t) - \frac{\pi}{2\gamma}\frac{\vartheta_2'\left(\frac{\pi t}{2\gamma}\right)}{\vartheta_2\left(\frac{\pi t}{2\gamma}\right)}}{\coth(\gamma-t)+\coth(\gamma+t)}, \qquad \mathfrak{s}(a,b,c) = \frac{\sinh(\gamma-t)\sinh(\gamma+t)}{\sinh(2\gamma)} \times$$

$$\times\sqrt{\frac{2}{3} - \frac{\pi^2}{12\gamma^2}\left(\frac{\vartheta_2'\left(\frac{\pi t}{2\gamma}\right)}{\vartheta_2\left(\frac{\pi t}{2\gamma}\right)}\right)^2 + \frac{\pi^2}{12\gamma^2}\sum_{j=1}^{4}\left(\frac{\vartheta_j'(\omega)}{\vartheta_j(\omega)}\right)^2 - \frac{\pi(\coth(\gamma+t)-\coth(\gamma-t))}{2\gamma}\frac{\vartheta_2'\left(\frac{\pi t}{2\gamma}\right)}{\vartheta_2\left(\frac{\pi t}{2\gamma}\right)} - \coth(\gamma+t)\coth(\gamma-t)}$$

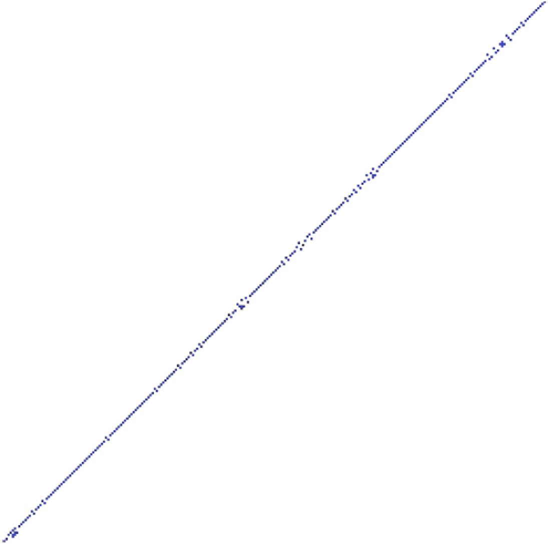
$\omega = \frac{\pi(1+t/\gamma)}{4}$, $\quad \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ = Jacobi elliptic theta functions with nome $q = e^{-\pi^2/(2\gamma)}$.

$\Delta > 1$: $N = 256$ simulation by David Keating

Is $\Delta < 1$ just a technical restriction?

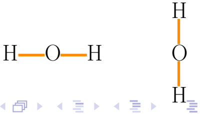
$\Delta > 1$: $N = 256$ simulation by David Keating

Is $\Delta < 1$ just a technical restriction? **No!**



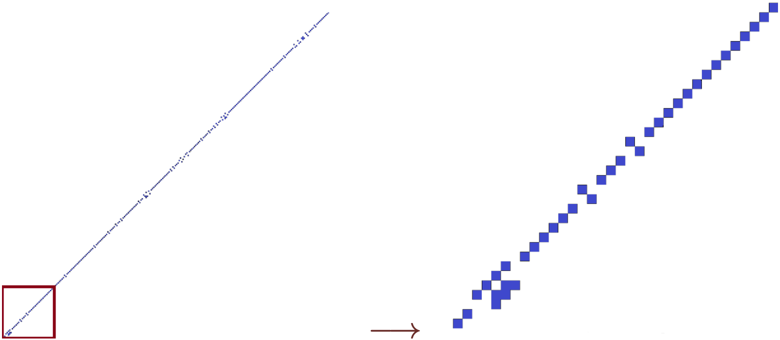
$a = 3$
$b = 1$
$c = 1$
$\Delta = \frac{3}{2}$

only c-vertices shown



$\Delta > 1$: stochastic six-vertex model.

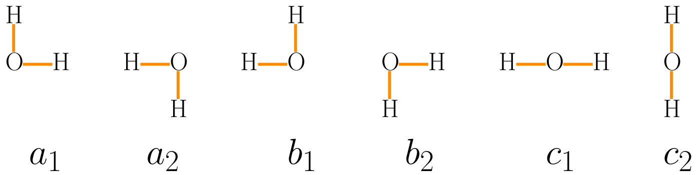
Theorem. (Gorin–Liechty-23) For $\Delta > 1$ and $a > b$, as $N \to \infty$ the configuration converges near the bottom–left corner to the **stochastic six-vertex model** *without any rescaling.*



(Complementary $a < b$ case is obtained by a vertical flip.)

Stochastic six–vertex model.

$$a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1.$$



Remark. This implies $\Delta = \frac{a_1 a_2 + b_1 b_2 - c_1 c_2}{2\sqrt{a_1 a_2 b_1 b_2}} \geq 1$.

The model in quadrant defined by **local sampling algorithm**.

Stochastic six–vertex model.

$$a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1.$$

The model in quadrant defined by **local sampling algorithm**.

⋮											
4	H	O	H	O	H	O	H	O	H	O	
		H		H		H		H		H	
3	H	O	H	O	H	O	H	O	H	O	
		H		H		H		H		H	
2	H	O	H	O	H	O	H	O	H	O	
		H		H		H		H		H	
1	H	O	H	O	H	O	H	O	H	O	
		1		2		3		4		5	…

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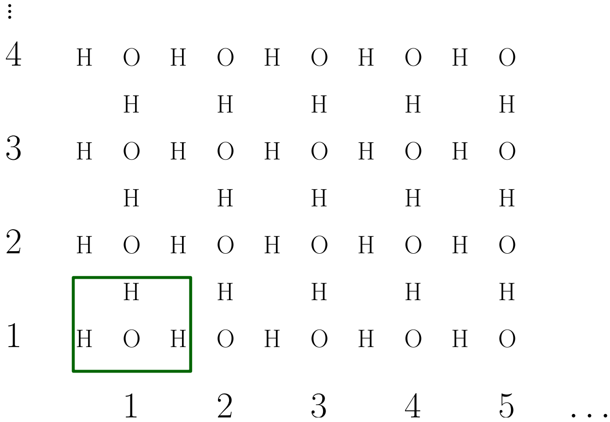

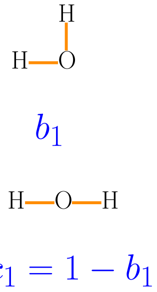

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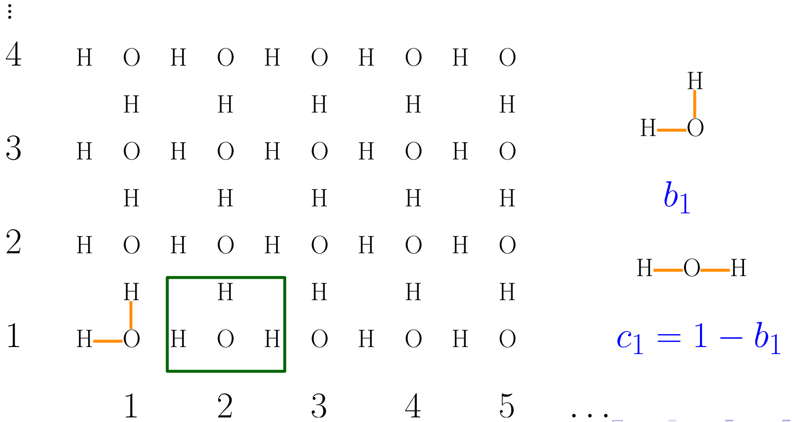

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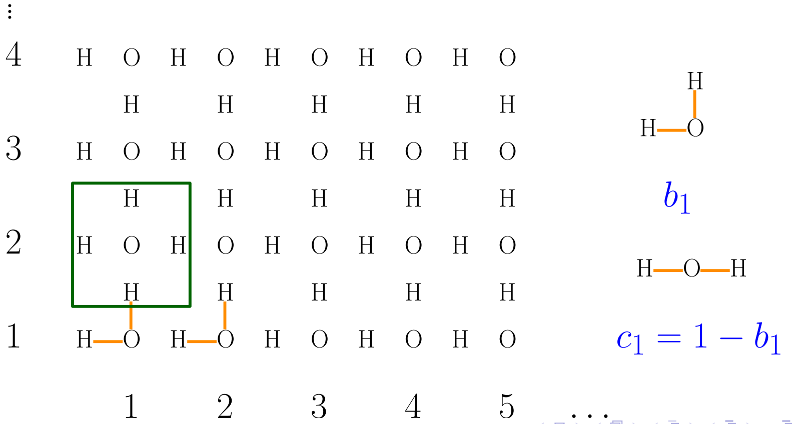

Stochastic six–vertex model.

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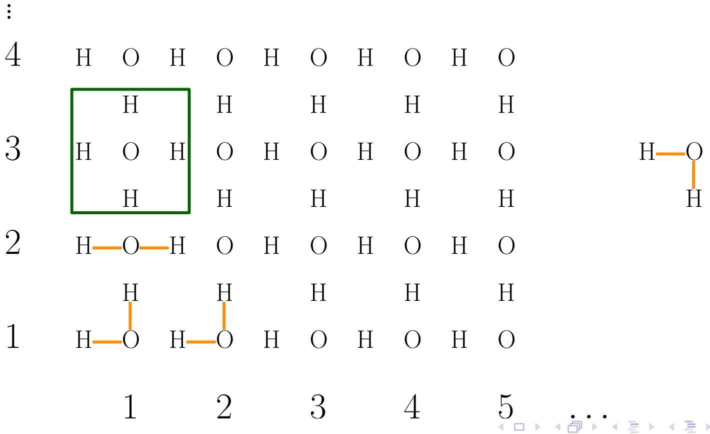
The model in quadrant defined by **local sampling algorithm**.



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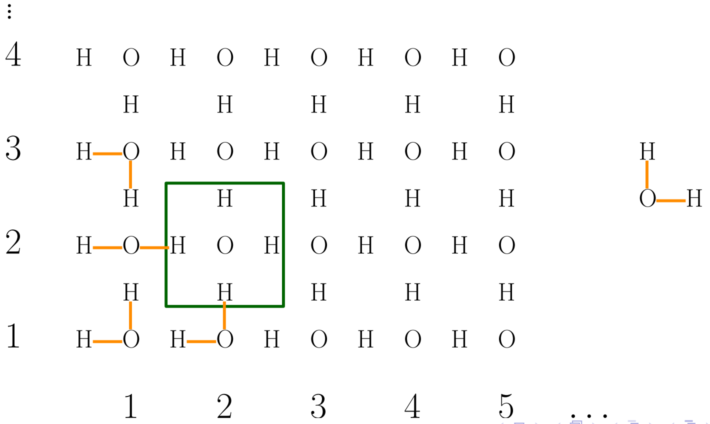

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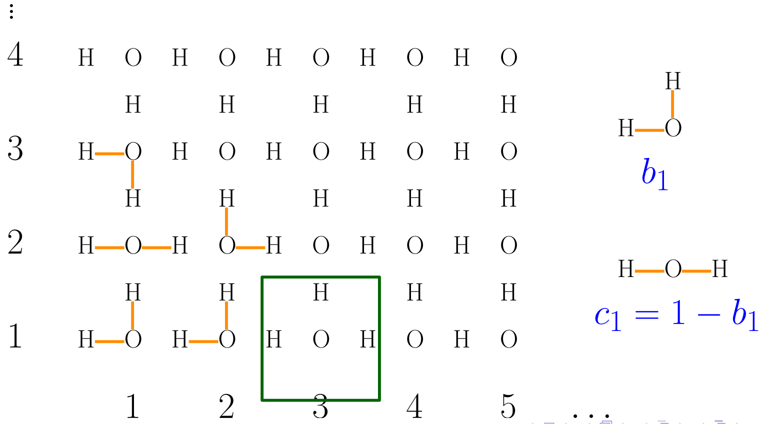

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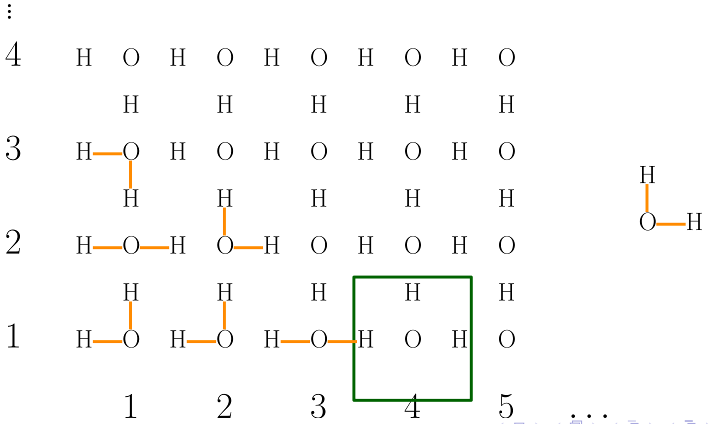

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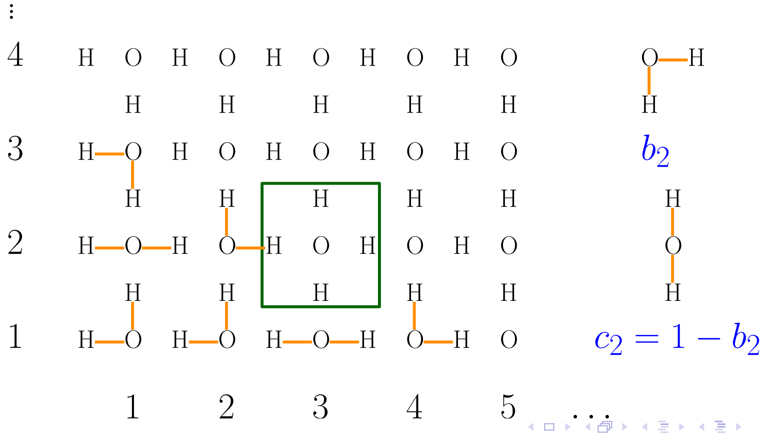

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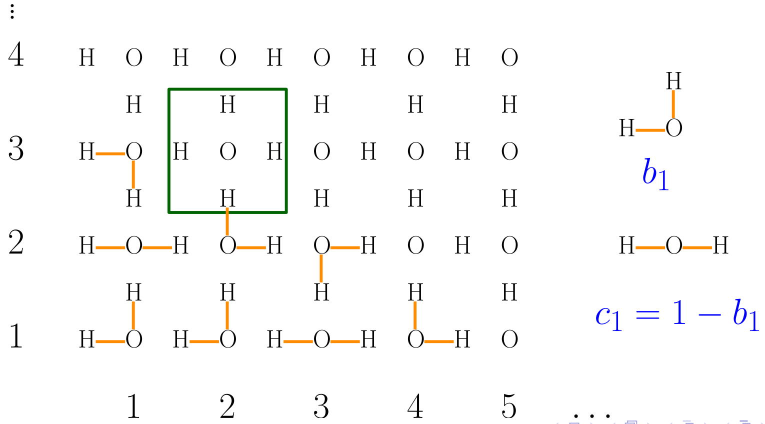

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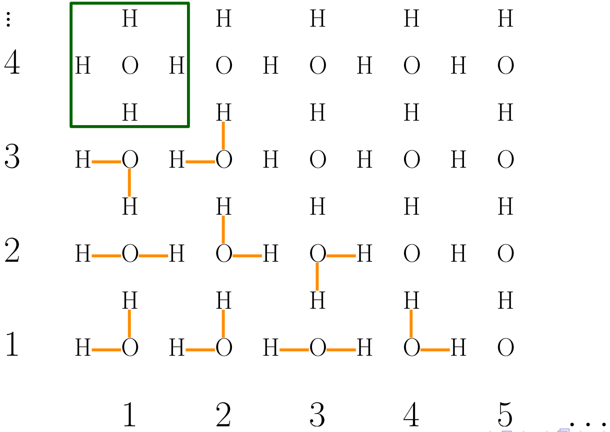

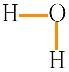

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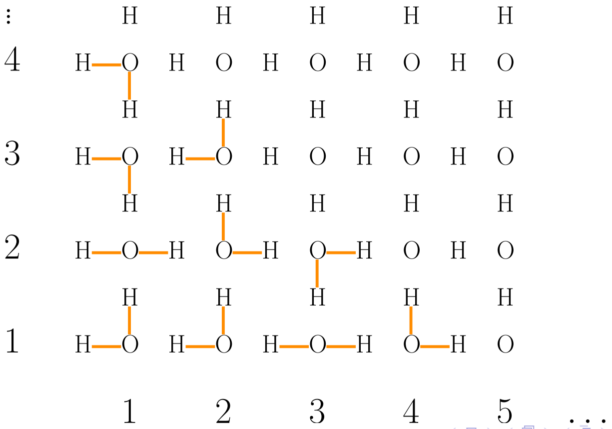

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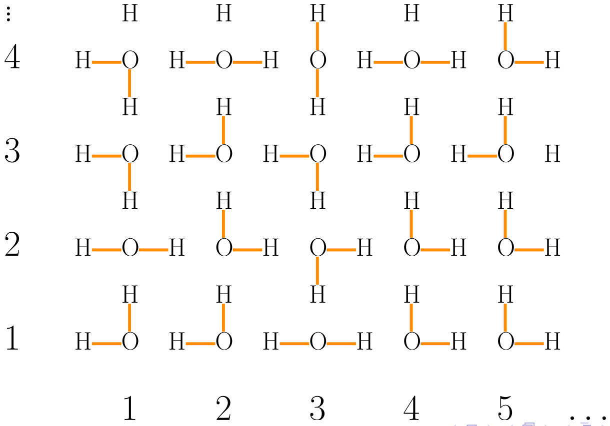
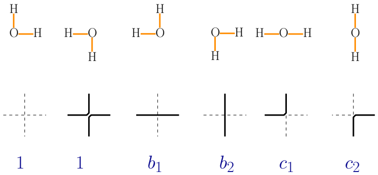
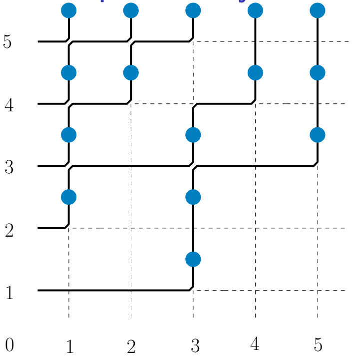


Stochastic six-vertex model is a particle system.



$$c_1 = 1 - b_1, \quad c_2 = 1 - b_2$$



- Discrete time version of Asymmetric Simple Exclusion Process.

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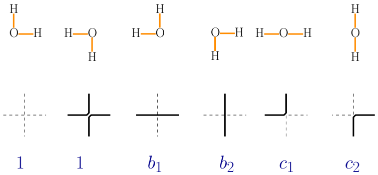

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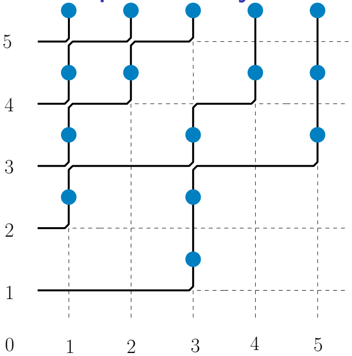

- Discrete time version of Asymmetric Simple Exclusion Process.
- First introduced on torus in [Gwa-Spohn-92].
- $b_1 > b_2$: LLN and fluctuations in [Borodin-Corwin-Gorin-16], [Dimitrov - 23]
  Translation-invariant case in [Aggarwal-18]
- Small $b_1 - b_2 > 0$ KPZ-limit in [Corwin-Ghosal-Shen-Tsai-20]
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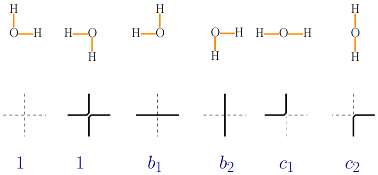

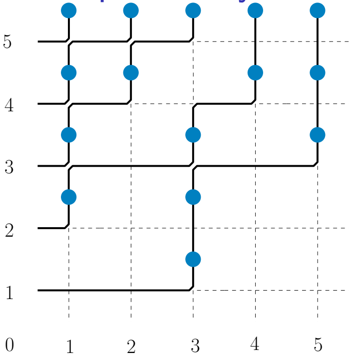

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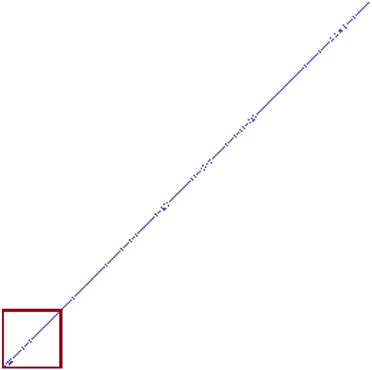
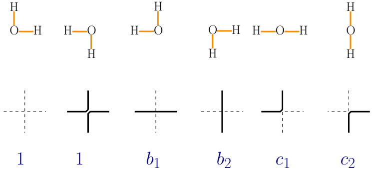
- Stationary regime $b_1 < b_2$ is relevant for DWBC.

$\Delta > 1$: stochastic six-vertex model.

Theorem. (Gorin–Liechty-23) For $\Delta > 1$ and $a > b$, as $N \to \infty$ the configuration converges near the bottom–left corner to the **stochastic six-vertex model** with $0 < b_1 < b_2 < 1$:

$$b_1 = \frac{a^2 + b^2 - c^2 - \sqrt{(a^2 + b^2 - c^2)^2 - 4a^2 b^2}}{2a^2}, \quad b_2 = \frac{a^2 + b^2 - c^2 + \sqrt{(a^2 + b^2 - c^2)^2 - 4a^2 b^2}}{2a^2}.$$





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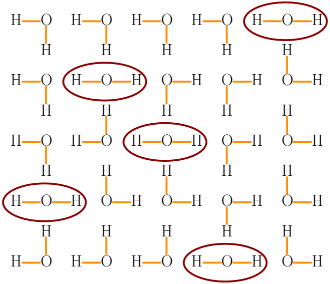

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For fixed N send $c \to 0$ to get the **Mallows measure on permutations**.

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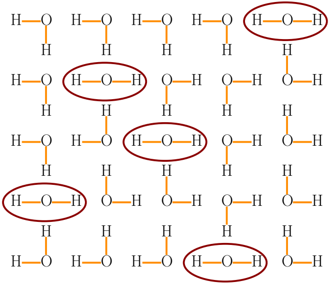

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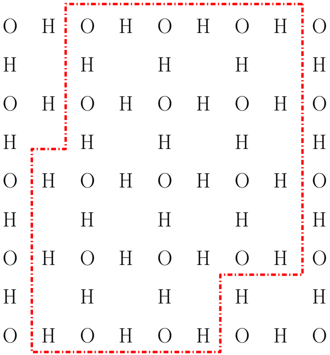
Proposition. Assume $a > b$ and $c = 0$. As $n \to \infty$ the permutation σ converges to **q–shuffle** of Gnedin and Olshanski: $\sigma(i)$ is $\mathrm{Geom}(q)$ largest element in $\mathbb{N} \setminus \{\sigma(1), \dots, \sigma(i-1)\}$.

$$q = \frac{b^2}{a^2}$$

General domains

Conjecture. For any $\Delta < 1$ and any large polygonal domain near boundaries we always see $\sqrt{N}$ fluctuations and GUE–eigenvalues.



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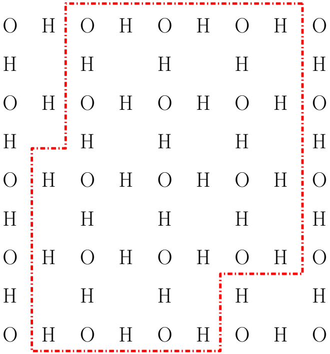

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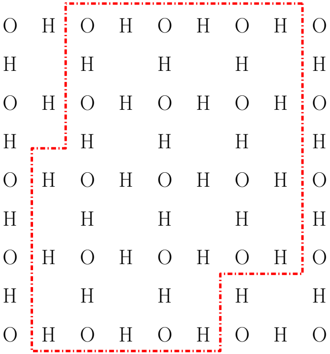

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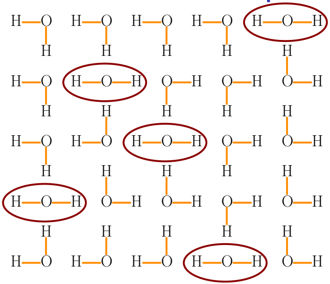
What about $\Delta \approx 1$? Teaser: Good simulations? How?

The simplest case to probe $\Delta \approx 1$.



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Proposition. Set $c = 0$, suppose $N \ln\left(\frac{b^2}{a^2}\right) \to \theta \in \mathbb{R}$ as $N \to \infty$. Then the rescaled by N positions of horizontal molecules converge in distribution to i.i.d. **truncated exponentials** of density

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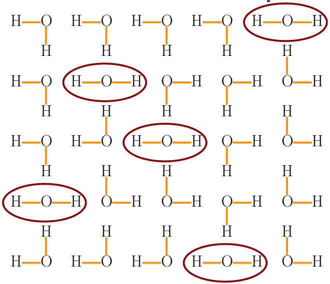

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Conclusion. We expect a rich world of boundary limits for $\varDelta \approx 1$.

A glimpse into proofs

Step 1. Introduce **row and column dependent** vertex weights.

$$\omega(x,y;\sigma) = \begin{cases} a(\psi_y - \chi_x, \gamma), \\ b(\psi_y - \chi_x, \gamma), \\ c(\gamma). \end{cases}$$

- **Ferroelectric phase**. For $\Delta > 1$

$$a(t,\gamma) = \sinh(t-\gamma), \quad b(t,\gamma) = \sinh(t+\gamma), \quad c(\gamma) = \sinh(2\gamma).$$

- **Disordered phase**. For $-1 < \Delta < 1$

$$a(t,\gamma) = \sin(\gamma - t), \quad b(t,\gamma) = \sin(\gamma + t), \quad c(\gamma) = \sin(2\gamma).$$

- **Antiferroelectric phase**. For $\Delta < -1$

$$a(t,\gamma) = \sinh(\gamma - t), \quad b(t,\gamma) = \sinh(\gamma + t), \quad c(\gamma) = \sinh(2\gamma).$$

- **Boundary phase.** For $\Delta = -1$,

$$a(t,\gamma) = \gamma - t, \quad b(t,\gamma) = \gamma + t, \quad c(\gamma) = 2\gamma.$$

$$\mathcal{Z}_N(\chi_1, \ldots, \chi_N; \psi_1, \ldots, \psi_N; \gamma) = \sum_\sigma \prod_{x=1}^N \prod_{y=1}^N \omega(x,y;\sigma).$$

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Step 2. *[Izergin, Korepin — 82, 87]* Partition function evaluates:

$$\frac{\prod_{i,j=1}^{N} \big(a(\psi_j-\chi_i,\gamma)b(\psi_j-\chi_i,\gamma)\big)}{\prod_{i<j}\big(b(\chi_i-\chi_j,0)b(\psi_i-\psi_j,0)\big)} \det\left[\frac{c(\gamma)}{a(\psi_j-\chi_i,\gamma)b(\psi_j-\chi_i,\gamma)}\right]_{i,j=1}^{N}.$$

Still open: Is there a structural explanation?

A glimpse into proofs

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Step 3. *[Gorin — 14]* The boundary limits can be read from

$$\frac{\mathcal{Z}_N(0^N; t + \xi_1, \ldots, t + \xi_k, t^{N-k}; \gamma)}{\mathcal{Z}_N(0^N; t^N; \gamma)}$$

How do we compute $N \to \infty$ asymptotics?

A glimpse into proofs

Step 1. $\mathcal{Z}_N(\chi_1,\ldots,\chi_N;\psi_1,\ldots,\psi_N;\gamma) = \sum_{\sigma}\prod_{x=1}^{N}\prod_{y=1}^{N}\omega(x,y;\sigma)$.

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Step 3. *[Gorin — 14]* Need $\dfrac{\mathcal{Z}_N(0^N;t+\xi_1,\ldots,t+\xi_k,t^{N-k};\gamma)}{\mathcal{Z}_N(0^N;t^N;\gamma)}$

Step 4. *[Zinn-Justin — 00]* **Laplace transform** helps:

$$\frac{c(\gamma)}{a(t,\gamma)b(t,\gamma)} = \int_{-\infty}^{\infty} e^{tx}\mathfrak{m}(\mathrm{d}x)$$

A glimpse into proofs

Theorem. Using **multivariate Bessel functions**

$$\mathcal{B}_{x_1,\dots,x_N}(z_1,\dots,z_N) = 1!2!\cdots(N-1)!\frac{\det\left[e^{x_i z_j}\right]_{i,j=1}^{N}}{\prod\limits_{1\le i<j\le N}(x_i-x_j)(z_i-z_j)}$$

and $\beta = 2$ **log-gas**,

$$\mathcal{M}^{N,t,\gamma} \sim \prod_{1\le i<j\le N}(x_i-x_j)^2\prod_{i=1}^{N}e^{tx_i}\mathfrak{m}(\mathrm{d}x_i),$$

we have

$$\frac{\mathcal{Z}_N(0^N;t+\xi_1,\dots,t+\xi_k,t^{N-k};\gamma)}{\mathcal{Z}_n(0^N;t^N;\gamma)}$$

$$=\prod_{j=1}^{k}\left[\left(\frac{a(t+\xi_j,\gamma)b(t+\xi_j,\gamma)}{a(t,\gamma)b(t,\gamma)}\right)^N\left(\frac{\xi_j}{b(\xi_j,0)}\right)^{N-k}\right]\prod_{i<j}\frac{\xi_i-\xi_j}{b(\xi_i-\xi_j,0)}$$

$$\times\,\mathbb{E}_{\mathcal{M}^{N,t,\gamma}}\left[\mathcal{B}_{x_1,\dots,x_N}(\xi_1,\dots,\xi_k,0^{N-k})\right].$$

A glimpse into proofs

An obstacle. The measure $\mathfrak{m}(\mathrm{d}x_i)$ is non-smooth.

1. For $\Delta > 1$, $\mathfrak{m}$ is supported on negative even integers:
$$\mathfrak{m} = \sum_{x \in 2\mathbb{Z}_{<0}} 2\sinh(-\gamma x)\,\delta_x;$$
2. For $-1 < \Delta < 1$, $\mathfrak{m}$ has density:
$$\mathfrak{m} = \frac{\sinh\left(\frac{x(\pi-2\gamma)}{2}\right)}{\sinh(\pi x/2)}\mathrm{d}x;$$
3. For $\Delta < -1$, $\mathfrak{m}$ is supported on even integers:
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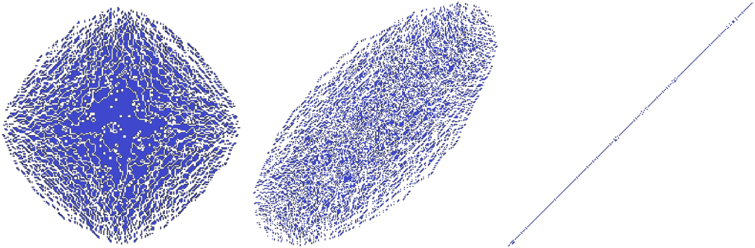
Our approach: through Riemann-Hilbert based analysis for the associated **orthogonal polynomials.**

Summary

Boundary limits for the 6v–model in $N \times N$ square with DWBC:

- **GUE** asymptotics after $\sqrt{N}$–rescaling for $\Delta < 1$.
- **Stationary stochastic six-vertex model** for $\Delta > 1$.
- Rich, but only partially understood limits for $\Delta \approx 1$.

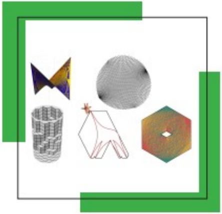


- Asymptotic analysis based on the Izergin-Korepin determinant.

ipam institute for pure & applied mathematics

Geometry, Statistical Mechanics, and Integrability

March 11 - June 14, 2024



Long Program Schedule

- Opening Day: March 11, 2024
- Geometry, Statistical Mechanics, and Integrability Tutorials: March 12-15, 2024
- Workshop I: Statistical Mechanics and Discrete Geometry: March 25-29, 2024
- Workshop II: Integrability and Algebraic Combinatorics: April 15-19, 2024
- Workshop III: Statistical Mechanics Beyond 2D: May 6-10, 2024
- Workshop IV: Vertex Models: Algebraic and Probabilistic Aspects of Universality: May 20-24, 2024
- Culminating Workshop at Lake Arrowhead: June 9-14, 2024

Organizers

Dmitry Chelkak (Uni of Michigan)
Jan de Gier (Univ. Melbourne)
Vadim Gorin (UC Berkeley)
Richard Kenyon (Yale)
Greta Panova (USC)
Sanjay Ramassamy (CNRS)
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Apply before October 11, 2023 at www.ipam.ucla.edu/gsi2024