Geometry, Statistical Mechanics, and Integrability

March 11 - June 14, 2024

Long Program Schedule

• Opening Day: March 11, 2024
• Geometry, Statistical Mechanics, and Integrability Tutorials: March 12-15, 2024
• Workshop I: Statistical Mechanics and Discrete Geometry: March 25-29, 2024
• Workshop II: Integrability and Algebraic Combinatorics: April 15-19, 2024
• Workshop III: Statistical Mechanics Beyond 2D: May 6-10, 2024
• Culminating Workshop at Lake Arrowhead: June 9-14, 2024

Organizers

Dmitry Chelkak (Uni of Michigan)
Jan de Gier (Univ. Melbourne)
Vadim Gorin (UC Berkeley)
Richard Kenyon (Yale)
Greta Panova (USC)
Sanjay Ramassamy (CNRS)
Marianna Russkikh (Caltech)

Apply before October 11, 2023 at www.ipam.ucla.edu/gsi2024
Boundary limits for the six-vertex model

Vadim Gorin

UC Berkeley

July, 2023
Six–vertex model

Square grid with $O$ in the vertices and $H$ on the edges.
Six–vertex model

Square grid with $O$ in the vertices and $H$ on the edges.

Take a finite/infinite domain.
Six–vertex model

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Take a finite/infinite domain.

*Configurations:* possible matchings of *all* atoms inside domain into $H_2O$ molecules.
Six–vertex model

Square grid with $O$ in the vertices and $H$ on the edges.

Take a finite/infinite domain.

**Configurations:** possible matchings of all atoms inside domain into $H_2O$ molecules.

This is **square ice model**. Real-world ice has somewhat similar (although 3d) structure.
Six–vertex model

Square grid with \(O\) in the vertices and \(H\) on the edges.

Take a finite/infinite domain.

**Configurations:** possible matchings of all atoms inside domain into \(H_2O\) molecules.

This is *square ice model*. Real-world ice has somewhat similar (although 3d) structure.
Square grid with $O$ in the vertices and $H$ on the edges.

**Configurations:** possible matchings of all atoms inside domain into $H_2O$ molecules.

This is **square ice model**. Real-world ice has somewhat similar (although 3d) structure.

Also known as the **six vertex model**.
Gibbs measures

Six positive weights corresponding to types of vertices.

\[
\begin{align*}
H & \quad H & \quad H & \quad H & \quad H & \quad H \\
O & \quad H & \quad O & \quad H & \quad O & \quad H & \quad O
\end{align*}
\]

\[
\begin{align*}
a_1 & \quad a_2 & \quad b_1 & \quad b_2 & \quad c_1 & \quad c_2
\end{align*}
\]

Remark. Distribution depends only on \(b_1\) and \(b_2\) and \(c_1\) and \(c_2\).

Example. Uniform measure on configurations in a fixed domain is Gibbs with \(a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1\).

We aim to study asymptotic properties of Gibbs measures.
Gibbs measures

Six positive weights corresponding to types of vertices.

\begin{align*}
\text{Gibbs} \text{ probability measure on configurations:} \\
\frac{a_1^\#(a_1) a_2^\#(a_2) b_1^\#(b_1) b_2^\#(b_2) c_1^\#(c_1) c_2^\#(c_2)}{Z(\Omega; a_1, a_2, b_1, b_2, c_1, c_2)}
\end{align*}

Remark. Distribution depends only on \(a_1\) \(a_2\) \(b_1\) \(b_2\) \(c_1\) \(c_2\).

Example. Uniform measure on configurations in a fixed domain is Gibbs with \(a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1\).

We aim to study asymptotic properties of Gibbs measures.
**Gibbs measures**

Six positive weights corresponding to types of vertices.

\[
\begin{align*}
&\text{H} \quad \text{H} \quad \text{H} \quad \text{H} \\
&\text{H} \quad \text{H} \quad \text{H} \quad \text{H} \\
&a_1 \quad a_2 \quad b_1 \quad b_2 \\
&\text{H} \quad \text{H} \quad \text{H} \quad \text{H} \\
&\text{H} \quad \text{H} \quad \text{H} \quad \text{H} \\
&\text{H} \quad \text{H} \quad \text{H} \quad \text{H} \\
&\text{H} \quad \text{H} \quad \text{H} \quad \text{H} \\
&\text{H} \quad \text{H} \quad \text{H} \quad \text{H} \\
&\text{H} \quad \text{H} \quad \text{H} \quad \text{H} \\
&\text{H} \quad \text{H} \quad \text{H} \quad \text{H} \\
&\text{H} \quad \text{H} \quad \text{H} \quad \text{H} \\
&\text{H} \quad \text{H} \quad \text{H} \quad \text{H} \\
\end{align*}
\]

**Gibbs** probability measure on configurations:

\[
\frac{a_1 \#(a_1) a_2 \#(a_2) b_1 \#(b_1) b_2 \#(b_2) c_1 \#(c_1) c_2 \#(c_2)}{Z(\Omega; a_1, a_2, b_1, b_2, c_1, c_2)}
\]

**Remark.** Distribution depends only on \(\frac{b_1 b_2}{a_1 a_2}\) and \(\frac{c_1 c_2}{a_1 a_2}\).
Gibbs measures

Gibbs probability measure on configurations:

\[
a_1^{\#(a_1)} a_2^{\#(a_2)} b_1^{\#(b_1)} b_2^{\#(b_2)} c_1^{\#(c_1)} c_2^{\#(c_2)} \over Z(\Omega; a_1, a_2, b_1, b_2, c_1, c_2)
\]

Remark. Distribution depends only on \( b_1 b_2 \over a_1 a_2 \) and \( c_1 c_2 \over a_1 a_2 \).

Example. *Uniform* measure on configurations in a fixed domain is Gibbs with \( a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1 \).
Gibbs measures

Gibbs probability measure on configurations:

\[
\frac{a_1^{\#(a_1)} a_2^{\#(a_2)} b_1^{\#(b_1)} b_2^{\#(b_2)} c_1^{\#(c_1)} c_2^{\#(c_2)}}{Z(\Omega; a_1, a_2, b_1, b_2, c_1, c_2)}
\]

Remark. Distribution depends only on \(\frac{b_1 b_2}{a_1 a_2}\) and \(\frac{c_1 c_2}{a_1 a_2}\).

Example. *Uniform* measure on configurations in a fixed domain is Gibbs with \(a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1\).

We aim to study asymptotic properties of Gibbs measures.
Domain wall boundary conditions (DWBC)

Simplest possible domain: $N \times N$ square.
Our setup: \((a, b, c)\)–measure with DWBC.

\[
N \times N \text{ square}
\]

\[
\begin{array}{cccccccc}
H & O & H & O & H & O & H & O \\
H & H & H & H & H & H & H & H \\
H & O & H & O & H & O & H & O \\
H & H & H & H & H & H & H & H \\
H & O & H & O & H & O & H & O \\
H & H & H & H & H & H & H & H \\
H & O & H & O & H & O & H & O \\
H & H & H & H & H & H & H & H \\
\end{array}
\]

\[
\text{Symmetric weights:}
\]

\[
\begin{array}{cccc}
\text{a} & \text{b} & \text{c} \\
\begin{array}{cccc}
H & O & H & O \\
H & H & H & H \\
H & O & H & O \\
H & H & H & H \\
\end{array} & \\
\begin{array}{cccc}
H & H & H & H \\
H & O & H & O \\
H & H & H & H \\
H & O & H & O \\
\end{array} & \\
\begin{array}{cccc}
H & H & H & H \\
H & O & H & O \\
H & H & H & H \\
H & O & H & O \\
\end{array} & \\
\begin{array}{cccc}
H & H & H & H \\
H & O & H & O \\
H & H & H & H \\
H & O & H & O \\
\end{array} \\
\end{array}
\]

No loss of generality, because of dependence on \(\frac{b_1 b_2}{a_1 a_2}\) and \(\frac{c_1 c_2}{a_1 a_2}\).

How does a random configuration look like as \(N \to \infty\)?

\[
\Delta = \frac{a^2 + b^2 - c^2}{2ab}
\]

will play a role.
Almost nothing in this picture was explained rigorously.
$N = 256$ simulation by David Keating

\[
\begin{align*}
 a &= 2 \\
 b &= 1 \\
 c &= 2 \\
 \Delta &= \frac{1}{4}
\end{align*}
\]

only $c$-vertices shown

\[
\text{H} - \text{O} - \text{H} \quad \text{H}
\]

\[
\text{H} - \text{O} - \text{H} \\
\text{H}
\]
• What happens near boundaries as $N \to \infty$?
• Boundary conditions are seen only through these points.
• By symmetries, it is sufficient to deal with lower boundary.
**GUE for all $\Delta < 1$**

**Theorem.** (Gorin–Liechty-23) For $\Delta < 1$ the probability that there are precisely $k$ horizontal molecules in line $k$ tends to 1 as $N \to \infty$. 

$$
\begin{array}{cccccccc}
H & O & H & O & H & O & H & O \\
H & H & H & H & H & H & H & H \\
H & O & H & O & H & O & H & O \\
H & H & H & H & H & H & H & H \\
H & O & H & O & H & O & H & O \\
H & H & H & H & H & H & H & H \\
\end{array}
$$

\[ \mathbb{N} \rightarrow \infty \]
Theorem. (Gorin–Liechty-23) For $\Delta < 1$, the positions of horizontal molecules in line $k$, after subtracting $m(a, b, c)N$ and dividing by $s(a, b, c)\sqrt{N}$, converge in distribution to the eigenvalues of $k \times k$ matrix of Gaussian Unitary Ensemble.

- Horizontal molecules uniquely fix all others.
- **Corollary:** The first $k$ rows $\to$ GUE–corners process.
**GUE for all $\Delta < 1$**

**Theorem.** (Gorin–Liechty-23) For $\Delta < 1$, the positions of horizontal molecules in line $k$, after subtracting $m(a, b, c)N$ and dividing by $s(a, b, c)\sqrt{N}$, converge in distribution to the eigenvalues of $k \times k$ matrix of Gaussian Unitary Ensemble.

![Diagram of horizontal molecules](image)

- Eigenvalues of $\frac{X+X^*}{2}$.
- $X = k \times k$ matrix with i.i.d. $\mathcal{N}(0, 1) + i\mathcal{N}(0, 1)$ elements.

- Horizontal molecules uniquely fix all others.
- **Corollary:** The first $k$ rows $\rightarrow$ GUE–corners process.
- Previous results:
  1. $\Delta = 0$: [Johansson-Nordenstam-06] through domino tilings.
GUE for all $\Delta < 1$

**Theorem.** (Gorin–Liechty-23) For $\Delta < 1$, the positions of horizontal molecules in line $k$, after subtracting $m(a, b, c)N$ and dividing by $s(a, b, c)\sqrt{N}$, converge in distribution to the eigenvalues of $k \times k$ matrix of Gaussian Unitary Ensemble.

$|\Delta| < 1$: $a = \sin(\gamma - t)$, $b = \sin(\gamma + t)$, $c = \sin(2\gamma)$, $|t| < \gamma < \pi/2$

$$m(a, b, c) = \frac{\cot(\gamma + t) + \frac{\pi}{2\gamma} \tan \left( \frac{\pi t}{2\gamma} \right)}{\cot(\gamma - t) + \cot(\gamma + t)}, \quad s(a, b, c) = \frac{\sin(\gamma - t) \sin(\gamma + t)}{\sin(2\gamma)} \times$$

$$\sqrt{\frac{2}{3} \left( \frac{\pi^2}{4\gamma^2} - 1 \right) - \left( \cot(\gamma - t) - \frac{\pi}{2\gamma} \tan \left( \frac{\pi t}{2\gamma} \right) \right) \left( \cot(\gamma + t) + \frac{\pi}{2\gamma} \tan \left( \frac{\pi t}{2\gamma} \right) \right)}.$$
GUE for all $\Delta < 1$

**Theorem.** (Gorin–Liechty-23) For $\Delta < 1$, the positions of horizontal molecules in line $k$, after subtracting $m(a, b, c)N$ and dividing by $s(a, b, c)\sqrt{N}$, converge in distribution to the eigenvalues of $k \times k$ matrix of Gaussian Unitary Ensemble.

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\[
m(a, b, c) = \frac{\cot(\gamma + t) + \frac{\pi}{2\gamma} \tan\left(\frac{\pi t}{2\gamma}\right)}{\cot(\gamma - t) + \cot(\gamma + t)}, \quad s(a, b, c) = \frac{\sin(\gamma - t) \sin(\gamma + t)}{\sin(2\gamma)} \times
\]
\[
\sqrt{\frac{2}{3} \left(\frac{\pi^2}{4\gamma^2} - 1\right) - \left(\cot(\gamma - t) - \frac{\pi}{2\gamma} \tan\left(\frac{\pi t}{2\gamma}\right)\right) \left(\cot(\gamma + t) + \frac{\pi}{2\gamma} \tan\left(\frac{\pi t}{2\gamma}\right)\right)}.
\]

$\Delta < -1$: $a = \sinh(\gamma - t), b = \sinh(\gamma + t), c = \sinh(2\gamma), |t| < \gamma$

\[
m(a, b, c) = \frac{\coth(\gamma + t) - \frac{\pi}{2\gamma} \vartheta_2'\left(\frac{\pi t}{2\gamma}\right)}{\coth(\gamma - t) + \coth(\gamma + t)}, \quad s(a, b, c) = \frac{\sinh(\gamma - t) \sinh(\gamma + t)}{\sinh(2\gamma)} \times
\]
\[
\sqrt{\frac{2}{3} - \frac{\pi^2}{12\gamma^2} \left(\vartheta_2'\left(\frac{\pi t}{2\gamma}\right)\right)^2 + \frac{\pi^2}{12\gamma^2} \sum_{j=1}^{4} \left(\vartheta_j'\left(\omega\right)\right)^2 - \frac{\pi (\coth(\gamma + t) - \coth(\gamma - t))}{2\gamma} \frac{\vartheta_2'\left(\frac{\pi t}{2\gamma}\right)}{\vartheta_2\left(\frac{\pi t}{2\gamma}\right)} - \coth(\gamma + t) \coth(\gamma - t)}
\]
\[
\omega = \frac{\pi(1 + t/\gamma)}{4}, \quad \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 = \text{Jacobi elliptic theta functions with nome } q = e^{-\pi^2/(2\gamma)}.\]
Δ > 1: \( N = 256 \) simulation by David Keating

Is \( \Delta < 1 \) just a technical restriction?
Δ > 1:  \( N = 256 \) simulation by David Keating

Is \( \Delta < 1 \) just a technical restriction? No!

\[
a = 3 \\
\Delta = \frac{3}{2}
\]

Only \( c \)-vertices shown

\[
\text{H--O--H} \quad \text{H--O--H}
\]
\( \Delta > 1: \) stochastic six-vertex model.

**Theorem.** (Gorin–Liechty-23) For \( \Delta > 1 \) and \( a > b \), as \( N \to \infty \) the configuration converges near the bottom–left corner to the **stochastic six-vertex model** without any rescaling.

(Complementary \( a < b \) case is obtained by a vertical flip.)
Stochastic six–vertex model.

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1. \]

**Remark.** This implies \( \Delta = \frac{a_1a_2 + b_1b_2 - c_1c_2}{2 \sqrt{a_1a_2b_1b_2}} \geq 1. \)

The model in quadrant defined by **local sampling algorithm**.
Stochastic six–vertex model.

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1. \]

The model in quadrant defined by \textbf{local sampling algorithm}.

\[ \begin{array}{cccccccc}
4 & H & O & H & O & H & O & H \\
H & H & H & H & H & H & & \\
3 & H & O & H & O & H & O & H & O \\
& H & H & H & H & H & H & \\
2 & H & O & H & O & H & O & H & O \\
& H & H & H & H & H & H & H & \\
1 & H & O & H & O & H & O & H & O & H & O \\
& 1 & 2 & 3 & 4 & 5 & \ldots & \\
\end{array} \]
Stochastic six–vertex model.

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1. \]

The model in quadrant defined by local sampling algorithm.

\[ b_1 \]

\[ c_1 = 1 - b_1 \]
Stochastic six–vertex model.

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The model in quadrant defined by **local sampling algorithm**.

\[ \begin{array}{ccccccc}
4 & H & O & H & O & H & O & H \\
    & H & H & H & H & H & H \\
3 & H & O & H & O & H & O & H \\
    & H & H & H & H & H & H \\
2 & H & O & H & O & H & O & H \\
    & H & H & H & H & H & H \\
1 & H & O & H & O & H & O & H \\
    & H & H & H & H & H & H \\
\end{array} \]

\[ b_1 \]

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Stochastic six–vertex model.

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1. \]

The model in quadrant defined by local sampling algorithm.

\[
\begin{align*}
4 & \quad \text{H O H O H O H O H O H O} \\
& \quad \text{H H H H H H H} \\
3 & \quad \text{H O H O H O H O H O H O} \\
& \quad \text{H H H H H H H} \\
2 & \quad \text{H O H O H O H O H O H O} \\
& \quad \text{H H H H H H H} \\
1 & \quad \text{H O H O H O H O H O H O} \\
& \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \ldots
\end{align*}
\]
Stochastic six–vertex model.

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1. \]

The model in quadrant defined by local sampling algorithm.
Stochastic six–vertex model.

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1. \]

The model in quadrant defined by local sampling algorithm.

\[ b_1 \quad \text{and} \quad c_1 = 1 - b_1 \]
Stochastic six–vertex model.

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1. \]

The model in quadrant defined by **local sampling algorithm**.
Stochastic six–vertex model.

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1. \]

The model in quadrant defined by local sampling algorithm.

\[ b_2 \]
\[ c_2 = 1 - b_2 \]
Stochastic six–vertex model.

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1. \]

The model in quadrant defined by local sampling algorithm.

\[ b_1 \]  \hspace{1cm}  \[ c_1 = 1 - b_1 \]
Stochastic six–vertex model.

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1. \]

The model in quadrant defined by local sampling algorithm.
Stochastic six–vertex model.

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1. \]

The model in quadrant defined by local sampling algorithm.
Stochastic six–vertex model.

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1. \]

The model in quadrant defined by local sampling algorithm.

\[
\begin{array}{ccccccc}
 & & & & & & \\
 & & & & & & \\
 & & & & & & \\
 & & & & & & \\
 & & & & & & \\
1 & 2 & 3 & 4 & 5 & \ldots &
\end{array}
\]
Stochastic six-vertex model is a particle system.

$c_1 = 1 - b_1, \quad c_2 = 1 - b_2$

- Discrete time version of Asymmetric Simple Exclusion Process.
Stochastic six-vertex model is a particle system.

\[ c_1 = 1 - b_1, \quad c_2 = 1 - b_2 \]

- Discrete time version of Asymmetric Simple Exclusion Process.
- First introduced on torus in [Gwa-Spohn-92].
- \( b_1 > b_2 \): LLN and fluctuations in [Borodin-Corwin-Gorin-16], [Dimitrov - 23]
- Translation-invariant case in [Aggarwal-18]
- Small \( b_1 - b_2 > 0 \) KPZ-limit in [Corwin-Ghosal-Shen-Tsai-20]
- Small \( b_1 - b_2 \) stochastic telegraph limit in [Borodin-Gorin-19], [Shen-Tsai-19]
Stochastic six-vertex model is a particle system.

\[ c_1 = 1 - b_1, \quad c_2 = 1 - b_2 \]

- Discrete time version of Asymmetric Simple Exclusion Process.
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- Small \( b_1 - b_2 \) stochastic telegraph limit in [Borodin-Gorin-19], [Shen-Tsai-19]
- Stationary regime \( b_1 < b_2 \) is relevant for DWBC.
\( \Delta > 1: \) stochastic six-vertex model.

**Theorem.** (Gorin–Liechty-23) For \( \Delta > 1 \) and \( a > b \), as \( N \to \infty \) the configuration converges near the bottom–left corner to the **stochastic six-vertex model** with \( 0 < b_1 < b_2 < 1 \):

\[
\begin{align*}
b_1 &= \frac{a^2 + b^2 - c^2 - \sqrt{(a^2 + b^2 - c^2)^2 - 4a^2b^2}}{2a^2}, \\
b_2 &= \frac{a^2 + b^2 - c^2 + \sqrt{(a^2 + b^2 - c^2)^2 - 4a^2b^2}}{2a^2}.
\end{align*}
\]

\( c_1 = 1 - b_1, \quad c_2 = 1 - b_2 \)
Special case: $c = 0$

For fixed $N$ send $c \to 0$ to get the Mallows measure on permutations.

$$\mathbb{P}(\sigma) \sim \left( \frac{b^2}{a^2} \right)^{\#\text{inversions}(\sigma)}$$

$$\Delta = \frac{a^2 + b^2}{2ab} \geq 1.$$
**Special case: \( c = 0 \)**

For fixed \( N \) send \( c \to 0 \) to get the Mallows measure on permutations.

\[
\mathbb{P}(\sigma) \sim \left( \frac{b^2}{a^2} \right)^{\#\text{inversions}(\sigma)}
\]

\[
\Delta = \frac{a^2 + b^2}{2ab} \geq 1.
\]

**Proposition.** Assume \( a > b \) and \( c = 0 \). As \( n \to \infty \) the permutation \( \sigma \) converges to \textit{q-shuffle} of Gnedin and Olshanski: \( \sigma(i) \) is Geom(\( q \)) largest element in \( \mathbb{N} \setminus \{ \sigma(1), \ldots, \sigma(i-1) \} \).

\[
q = \frac{b^2}{a^2}
\]
**General domains**

**Conjecture.** For any $\Delta < 1$ and any large polygonal domain near boundaries we always see $\sqrt{N}$ fluctuations and GUE–eigenvalues.

- We proved it for squares.
- [Aggarwal–Gorin-22] An analogue for lozenge tilings $\approx$ five-vertex model.
**General domains**

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**Open question.** What are all possible boundary limits for $\Delta > 1$?

We found stationary stochastic six–vertex model.

[Dimitrov-20, Dimitrov-Rychnovsky-22] Some infinite domains $\rightarrow$ GUE.
**General domains**

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- We found stationary stochastic six–vertex model.
- [Dimitrov-20, Dimitrov-Rychnovsky-22] Some infinite domains $\rightarrow$ GUE.

What about $\Delta \approx 1$? **Teaser:** Good simulations? How?
The simplest case to probe $\Delta \approx 1$.

For fixed $N$ send $c \to 0$ to get the Mallows measure on permutations.

$$\mathbb{P}(\sigma) \sim \left( \frac{b^2}{a^2} \right)^{\#\text{inversions}(\sigma)}$$

$$\Delta = \frac{a^2 + b^2}{2ab} \geq 1.$$

**Proposition.** Set $c = 0$, suppose $N \ln \left( \frac{b^2}{a^2} \right) \to \theta \in \mathbb{R}$ as $N \to \infty$. Then the rescaled by $N$ positions of horizontal molecules converge in distribution to i.i.d. **truncated exponentials** of density

$$\rho_\eta(x) = \frac{\theta}{e^\theta - 1} e^{\theta x}, \quad x \in [0, 1].$$
The simplest case to probe $\Delta \approx 1$.

For fixed $N$ send $c \to 0$ to get the Mallows measure on permutations.

$\mathbb{P}(\sigma) \sim \left( \frac{b^2}{a^2} \right)^{\# \text{inversions}(\sigma)}$

$\sigma = 41325$

$\Delta = \frac{a^2 + b^2}{2ab} \geq 1.$

**Proposition.** Set $c = 0$, suppose $N \ln \left( \frac{b^2}{a^2} \right) \to \theta \in \mathbb{R}$ as $N \to \infty$. Then the rescaled by $N$ positions of horizontal molecules converge in distribution to i.i.d. truncated exponentials of density

$$\rho_\eta(x) = \frac{\theta}{e^\theta - 1} e^{\theta x}, \quad x \in [0, 1].$$

**Conclusion.** We expect a rich world of boundary limits for $\Delta \approx 1$. $
$
A glimpse into proofs

Step 1. Introduce row and column dependent vertex weights.

\[ \omega(x, y; \sigma) = \begin{cases} 
  a(\psi_y - \chi_x, \gamma), \\
  b(\psi_y - \chi_x, \gamma), \\
  c(\gamma).
\end{cases} \]

- **Ferroelectric phase.** For \( \Delta > 1 \)
  \[ a(t, \gamma) = \sinh(t - \gamma), \quad b(t, \gamma) = \sinh(t + \gamma), \quad c(\gamma) = \sinh(2\gamma). \]

- **Disordered phase.** For \(-1 < \Delta < 1\)
  \[ a(t, \gamma) = \sin(\gamma - t), \quad b(t, \gamma) = \sin(\gamma + t), \quad c(\gamma) = \sin(2\gamma). \]

- **Antiferroelectric phase.** For \( \Delta < -1 \)
  \[ a(t, \gamma) = \sinh(\gamma - t), \quad b(t, \gamma) = \sinh(\gamma + t), \quad c(\gamma) = \sinh(2\gamma). \]

- **Boundary phase.** For \( \Delta = -1 \),
  \[ a(t, \gamma) = \gamma - t, \quad b(t, \gamma) = \gamma + t, \quad c(\gamma) = 2\gamma. \]

\[ Z_N(\chi_1, \ldots, \chi_N; \psi_1, \ldots, \psi_N; \gamma) = \sum_{\sigma} \prod_{x=1}^{N} \prod_{y=1}^{N} \omega(x, y; \sigma). \]
A glimpse into proofs

Step 1. Introduce row and column dependent vertex weights.

\[ \omega(x, y; \sigma) = \begin{cases} 
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  b(\psi_y - \chi_x, \gamma), \\
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\end{cases} \]

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Step 2. [Izergin, Korepin — 82, 87] Partition function evaluates:

\[ \frac{\prod_{i,j=1}^{N} (a(\psi_j - \chi_i, \gamma)b(\psi_j - \chi_i, \gamma))}{\prod_{i<j} (b(\chi_i - \chi_j, 0)b(\psi_i - \psi_j, 0))} \det \left[ \frac{c(\gamma)}{a(\psi_j - \chi_i, \gamma)b(\psi_j - \chi_i, \gamma)} \right]_{i,j=1}^{N}. \]

Still open: Is there a structural explanation?
A glimpse into proofs

Step 1. Introduce row and column dependent vertex weights.

\[ \omega(x, y; \sigma) = \begin{cases} 
  a(\psi_y - \chi_x, \gamma), \\
  b(\psi_y - \chi_x, \gamma), \\
  c(\gamma). 
\end{cases} \]

\[ Z_N(\chi_1, \ldots, \chi_N; \psi_1, \ldots, \psi_N; \gamma) = \sum_{\sigma} \prod_{x=1}^{N} \prod_{y=1}^{N} \omega(x, y; \sigma). \]

Step 2. [Izergin, Korepin — 82, 87] Partition function evaluates:

\[ \prod_{i,j=1}^{N} (a(\psi_j - \chi_i, \gamma)b(\psi_j - \chi_i, \gamma)) \frac{\prod (b(\chi_i - \chi_j, 0)b(\psi_i - \psi_j, 0))}{\prod (b(\chi_i - \chi_j, 0)b(\psi_i - \psi_j, 0))} \det \left[ \begin{array}{c} c(\gamma) \\
  a(\psi_j - \chi_i, \gamma)b(\psi_j - \chi_i, \gamma) \end{array} \right]_{i,j=1}^{N}. \]

Step 3. [Gorin — 14] The boundary limits can be read from

\[ \frac{Z_N(0^N; t + \xi_1, \ldots, t + \xi_k, t^{N-k}; \gamma)}{Z_N(0^N; t^{N}; \gamma)}. \]

How do we compute \( N \to \infty \) asymptotics?
A glimpse into proofs

Step 1. \( \mathcal{Z}_N(\chi_1, \ldots, \chi_N; \psi_1, \ldots, \psi_N; \gamma) = \sum_{\sigma} \prod_{x=1}^{N} \prod_{y=1}^{N} \omega(x, y; \sigma). \)

Step 2. [Izergin, Korepin — 82, 87] Partition function evaluates:

\[
\prod_{i,j=1}^{N} \frac{(a(\psi_j - \chi_i, \gamma)b(\psi_j - \chi_i, \gamma))}{\prod_{i<j}(b(\chi_i - \chi_j, 0)b(\psi_i - \psi_j, 0))} \det \left[ \frac{c(\gamma)}{a(\psi_j - \chi_i, \gamma)b(\psi_j - \chi_i, \gamma)} \right]_{i,j=1}^{N}.
\]

Step 3. [Gorin — 14] Need \( \frac{\mathcal{Z}_N(0^N; t + \xi_1, \ldots, t + \xi_k, t^{N-k}; \gamma)}{\mathcal{Z}_N(0^N; t^N; \gamma)} \)

Step 4. [Zinn-Justin — 00] Laplace transform helps:

\[
\frac{c(\gamma)}{a(t, \gamma)b(t, \gamma)} = \int_{-\infty}^{\infty} e^{tx}m(dx)
\]
A glimpse into proofs

**Theorem.** Using **multivariate Bessel functions**

\[ B_{x_1,\ldots,x_N}(z_1,\ldots,z_N) = 1!2!\cdots(N-1)! \frac{\det[e^{x_i z_j}]_{i,j=1}^N}{\prod_{1\leq i<j\leq N} (x_i-x_j)(z_i-z_j)} \]

and \( \beta = 2 \) **log-gas**, we have

\[ \mathcal{M}^{N,t,\gamma} \sim \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{i=1}^N e^{tx_i} m(dx_i), \]

we have

\[
\frac{Z_N(0^N; t + \xi_1, \ldots, t + \xi_k, t^{N-k}; \gamma)}{Z_(0^N; t^{N}; \gamma)} = \prod_{j=1}^k \left[ \left( \frac{a(t + \xi_j, \gamma)b(t + \xi_j, \gamma)}{a(t, \gamma)b(t, \gamma)} \right)^N \left( \frac{\xi_j}{b(\xi_j, 0)} \right)^{N-k} \right] \prod_{i<j} \frac{\xi_i - \xi_j}{b(\xi_i - \xi_j, 0)}
\]

\[ \times \mathbb{E}_{\mathcal{M}^{N,t,\gamma}} [B_{x_1,\ldots,x_N}(\xi_1,\ldots,\xi_k,0^{N-k})]. \]
A glimpse into proofs

**An obstacle.** The measure $m(dx_i)$ is non-smooth.

1. For $\Delta > 1$, $m$ is supported on negative even integers:
   \[ m = \sum_{x \in 2\mathbb{Z} < 0} 2 \sinh(-\gamma x) \delta_x; \]

2. For $-1 < \Delta < 1$, $m$ has density:
   \[ m = \frac{\sinh \left( \frac{x(\pi - 2\gamma)}{2} \right)}{\sinh(\pi x/2)} dx; \]

3. For $\Delta < -1$, $m$ is supported on even integers:
   \[ m = \sum_{x \in 2\mathbb{Z}} 2e^{-\gamma|x|} \delta_x; \]

4. For $-\Delta = -1$, $m$ has density:
   \[ m = e^{-|x|} dx. \]
A glimpse into proofs

An obstacle. The measure $m(dx_i)$ is non-smooth.

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$$m = e^{-|x|} dx.$$

Our approach: through Riemann-Hilbert based analysis for the associated orthogonal polynomials.
Summary

Boundary limits for the 6v–model in $N \times N$ square with DWBC:

- **GUE** asymptotics after $\sqrt{N}$–rescaling for $\Delta < 1$.
- **Stationary stochastic six-vertex model** for $\Delta > 1$.
- Rich, but only partially understood limits for $\Delta \approx 1$.

- Asymptotic analysis based on the Izergin-Korepin determinant.
Geometry, Statistical Mechanics, and Integrability

March 11 - June 14, 2024

Long Program Schedule

- Opening Day: March 11, 2024
- Geometry, Statistical Mechanics, and Integrability Tutorials: March 12-15, 2024
- Workshop I: Statistical Mechanics and Discrete Geometry: March 25-29, 2024
- Workshop II: Integrability and Algebraic Combinatorics: April 15-19, 2024
- Workshop III: Statistical Mechanics Beyond 2D: May 6-10, 2024
- Culminating Workshop at Lake Arrowhead: June 9-14, 2024

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