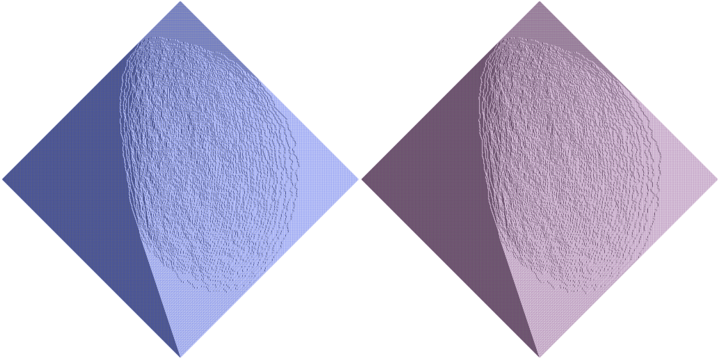
Coupled tilings, LLT polynomials, and double dimers

David Keating
University of Wisconsin, Madison

DIMERS ANR Final Conference

Outline

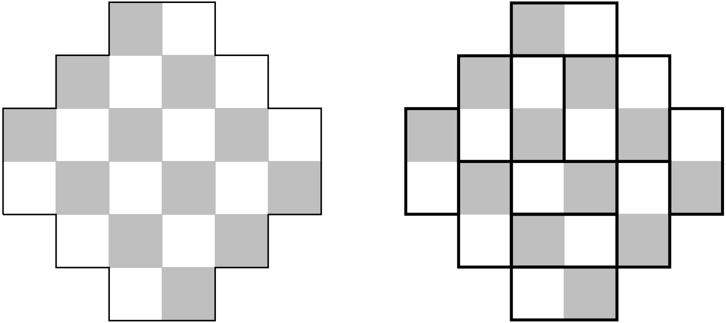
1. Overview of tilings of the Aztec diamond
2. Defining the coupled tilings (based on work with Sylvie Corteel and Andrew Gitlin: arXiv:2202.06020)
3. Simulations
4. A shuffling algorithm (based on work with Matthew Nicoletti: arXiv:2303.09089)



Part 1: Review of the Aztec diamond

Domino tilings of the Aztec diamond

Domino tilings of the Aztec diamond were first introduced by Elkies, Kuperberg, Larsen, and Propp in 1992.

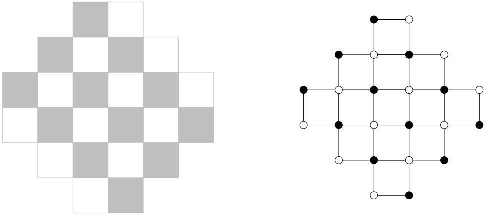


The Aztec diamond of rank $m = 3$ and one possible domino tiling.

Domino tilings of the Aztec diamond

There are many ways to view these tilings:

- As a dimer model

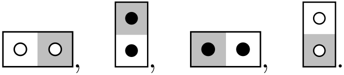


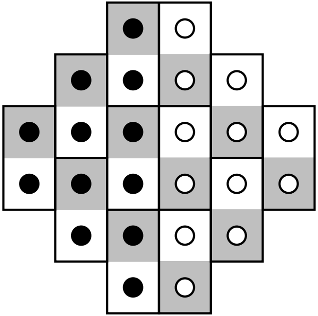
- As an example of a Schur process.
- As an integrable vertex model.

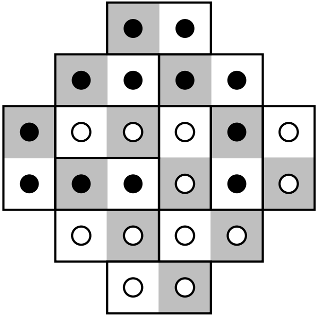
For the moment we'll focus on the the last two points.

Domino tilings and sequences of partitions

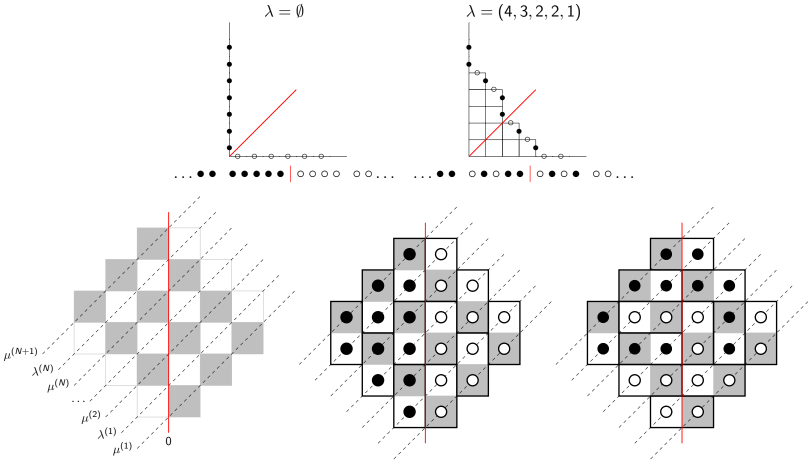
Assign ‘particles’ and ‘holes’ to our dominos according to the rules





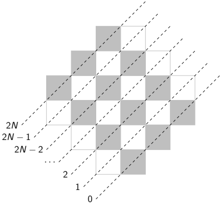


Domino tilings and sequences of partitions



$$\emptyset = \mu^{(1)} \preceq \lambda^{(1)} \succeq' \mu^{(2)} \preceq \ldots \preceq \lambda^{(N-1)} \succeq' \mu^{(N)} \preceq \lambda^{(N)} \succeq' \mu^{(N+1)} = \emptyset$$

Weights



Assign weights to the dominos according to:

- A  domino whose left square is on slice $2i-1$ gets a weight of x_i.
- A  domino whose right square is on slice $2i-1$ gets a weight of y_i.
- All other dominos get weight of 1.

Then the weight of a tiling

$$\emptyset \preceq \lambda^{(1)} \succeq' \mu^{(2)} \preceq \ldots \preceq \lambda^{(N-1)} \succeq' \mu^{(N)} \preceq \lambda^{(N)} \succeq' \emptyset$$

can be written as

$$s_{\lambda^{(1)}}(x_1) s_{(\lambda^{(1)}/\mu^{(2)})'}(y_1) s_{\lambda^{(2)}/\mu^{(2)}}(x_2) \ldots s_{\lambda^{(N)}/\mu^{(N)}}(x_N) s_{(\lambda^{(N)})'}(y_N)$$

Enumeration

Repeated applications of the Cauchy identity

$$\sum_{\lambda} s_{\lambda/\nu}(X) s_{\lambda'/\mu'}(Y) = \left(\prod_{i,j} (1 + x_i y_j) \right) \sum_{\lambda} s_{\nu'/\lambda'}(Y) s_{\mu/\lambda}(X) \ .$$

and branching rule

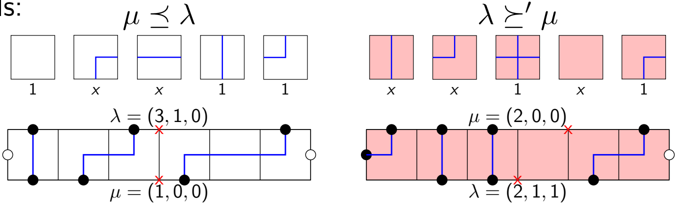
$$\sum_{\mu} s_{\lambda/\mu}(X) s_{\mu}(Y) = s_{\lambda}(X, Y)$$

can be used to show

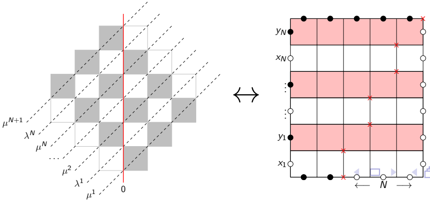
$$Z_{AD}(X, Y) = \prod_{i \leq j} (1 + x_i y_j)$$

Domino tilings as an integrable vertex model

Equivalently, one can view the tilings in terms of integrable vertex models:

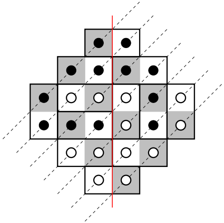


There is a weight-preserving bijection between tiling (as a sequence of partitions) and vertex model

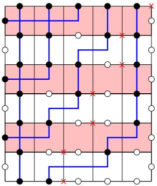


Domino tilings as an integrable vertex model

There is a weight-preserving bijection between tiling (as a sequence of partitions) and vertex model



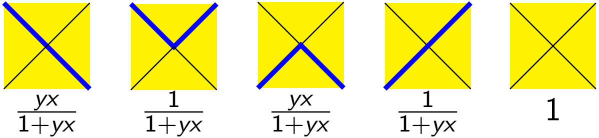
$\leftrightarrow$

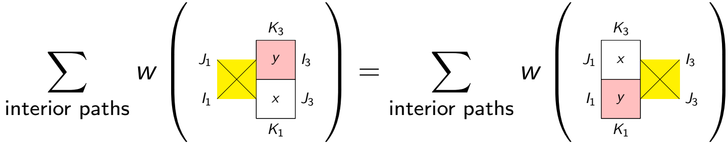


$$x_1^2 x_2 x_3 y_2^2 y_3^3 \quad \leftrightarrow \quad \underbrace{(y_1^3 y_2^2 y_3^1)}_{\text{ind. of configuration}} \quad x_1^2 x_2 x_3 y_2^2 y_3^3$$

Domino tilings as an integrable vertex model

These vertex models satisfy the Yang-Baxter equation:

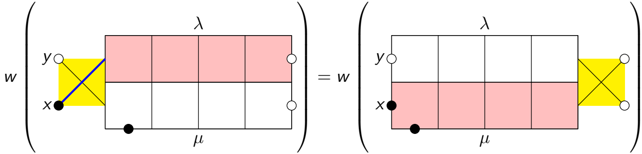




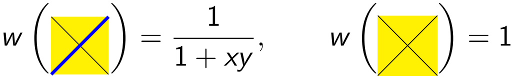
for any fixed choice of boundary condition $I_1, J_1, K_1, I_3, J_3, K_3$.

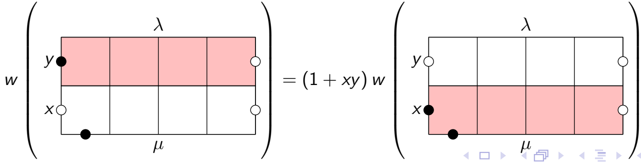
Domino tilings as an integrable vertex model

We can repeatedly apply the YBE to swap rows of the vertex model:

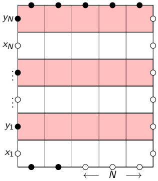


Then removing the yellow faces (but keeping the weight) gives

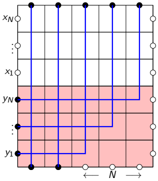




Domino tilings as an integrable vertex model



$$= \text{many row swaps} = \left(\prod_{i \le j} (1 + x_i y_j) \right)$$



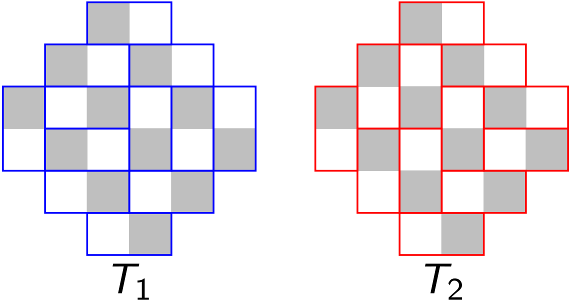
only config.
weight: $y_1^N y_2^{N-1} \dots y_N$

$$\implies Z_{AD}(X, Y) = \prod_{i \le j} (1 + x_i y_j)$$

Part 2: Coupled tilings of the Aztec diamond

Coupled tilings

Now rather than a single tiling we will consider a pair of tilings:



We'll refer to the tilings as being different colors. We order the colors blue $<$ red.

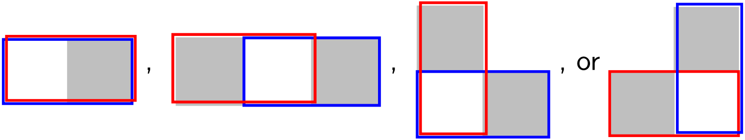
Weights of the coupled tiling

Assign weights to the dominos according to the rules

- A domino of the form  whose left square is on slice $2i - 1$ gets a weight of x_i.
- A domino of the form  whose right square is on slice $2i - 1$ gets a weight of y_i.
- All other dominos get a weight of 1.

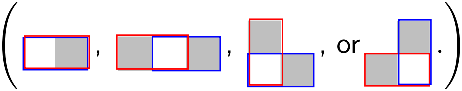
for each color.

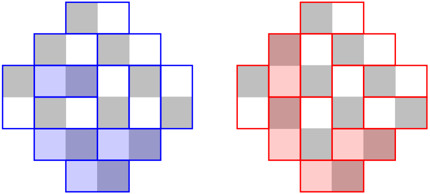
Each 'interaction' gives a power of t, $t \geq 0$, where we define 'interaction' by



Weights of the coupled tiling

In our example,



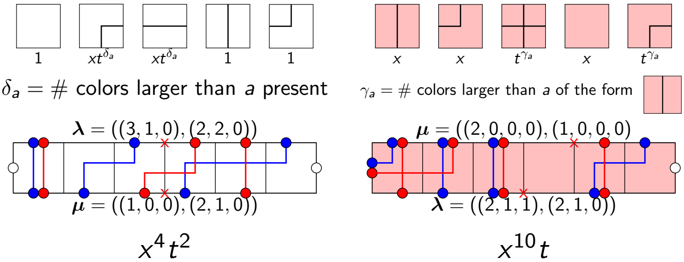


T_1 T_2

which has weight $\underbrace{x_1^2 x_2 y_2^2 x_3 y_3^2 x_1^3 y_1 y_2 y_3}_{\text{from hor. dominos}} \underbrace{t^4}_{\text{interactions}}$.

Where do the weights come from?

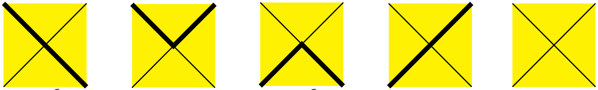
If we superimpose the two copies of our five-vertex models, we get a new colored vertex model



These vertex models are a degeneration of a vertex model studied by Aggarwal, Borodin, and Wheeler (2021) related to the quantum group $U_q(\hat{\mathfrak{sl}}(1|k))$.

Enumeration

The colored vertex model is still Yang-Baxter integrable (inherited from the vertex model of Aggarwal-Borodin-Wheeler, see also Corteel-Gitlin-K.-Meza 2020)



$\frac{yxt^{\epsilon_a}}{1+yxt^{\epsilon_a}}$ $\frac{1}{1+yxt^{\epsilon_a}}$ $\frac{yxt^{\epsilon_a}}{1+yxt^{\epsilon_a}}$ $\frac{1}{1+yxt^{\epsilon_a}}$ 1

$\epsilon_a = \#$ colors larger than a present

Using the integrability exactly as before, we have

Theorem (Corteel-Gitlin-K. 2022)

The partition function for the coupled tilings of the Aztec diamond is given by

$$Z_{AD}^{(2)}(X, Y; t) = \prod_{i \leq j} (1 + x_i y_j)(1 + x_i y_j t)$$

Where do the weights come from?

In terms of partitions we now have a bijection between tilings and sequences of 2-tuples of interlacing partitions.

$$\emptyset \preceq \underbrace{\boldsymbol{\lambda}^{(1)}}_{=(\lambda^{(1)},\lambda^{(1)})} \succeq' \boldsymbol{\mu}^{(2)} \preceq \dots \preceq \boldsymbol{\lambda}^{(N-1)} \succeq' \boldsymbol{\mu}^{(N)} \preceq \boldsymbol{\lambda}^{(N)} \succeq' \emptyset$$

The weight of the tiling can be written as

$$t^{\#}\mathcal{L}_{\boldsymbol{\lambda}^{(1)}}(x_1;t)\tilde{\mathcal{L}}_{\boldsymbol{\lambda}^{(1)}/\boldsymbol{\mu}^{(2)}}(y_1;t)\mathcal{L}_{\boldsymbol{\lambda}^{(2)}/\boldsymbol{\mu}^{(2)}}(x_2;t)\tilde{\mathcal{L}}_{\boldsymbol{\lambda}^{(2)}/\boldsymbol{\mu}^{(3)}}(y_2;t)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(N)}/\boldsymbol{\mu}^{(N)}}(x_N;t)\tilde{\mathcal{L}}_{\boldsymbol{\lambda}^{(N)}}(y_N;t)$$

The $\mathcal{L}$ are called LLT polynomials and are a generalization of the Schur polynomials.

Remarks

- Everything here makes sense for more than 2 colors. Interactions are then counted between every pair of colors.

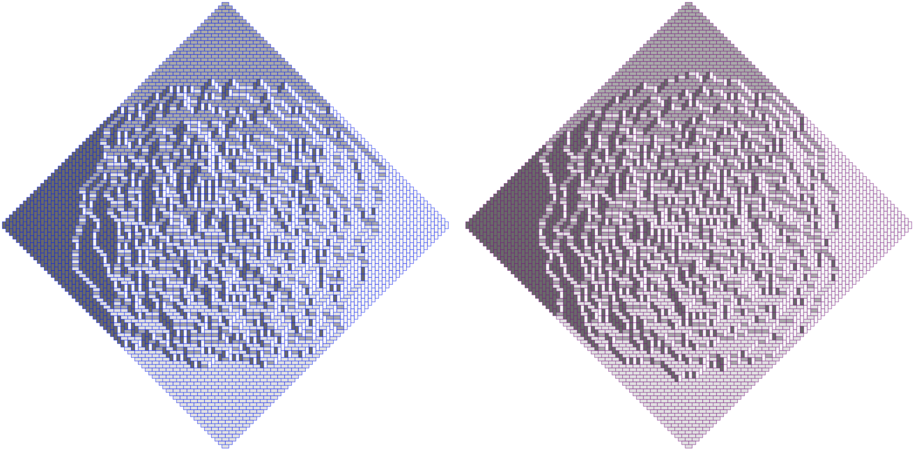
$$k \text{ colors: } Z_{AD}^{(k)}(X, Y; t) = \prod_{\ell=0}^{k-1} \prod_{i \leq j} (1 + x_i y_j t^{\ell})$$

- Similar constructions can be done for other examples of types of tilings. For example, reverse plane partitions.

$$Z_{RPP,\lambda}^{(k)}(q; t) = \prod_{\ell=0}^{k-1} \prod_{u \in \lambda} \frac{1}{1 - q^{h(u)} t^{\ell}}$$

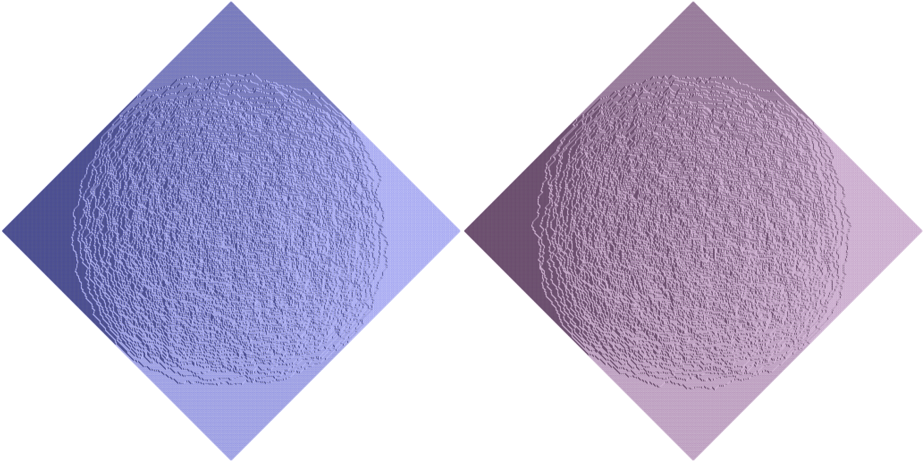
Part 3: Simulations

Simulations



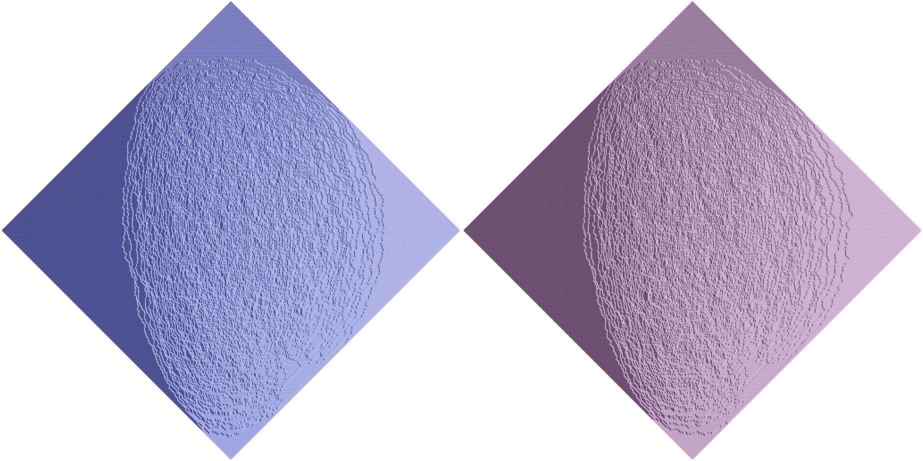
Simulation of a 2-tiling of the rank-64 Aztec diamond at $t = 1$.

Simulations



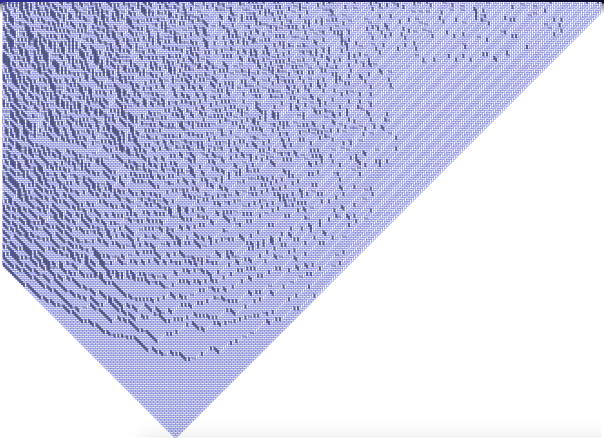
Simulation of a 2-tiling of the rank-256 Aztec diamond at $t = 1$.

Simulations



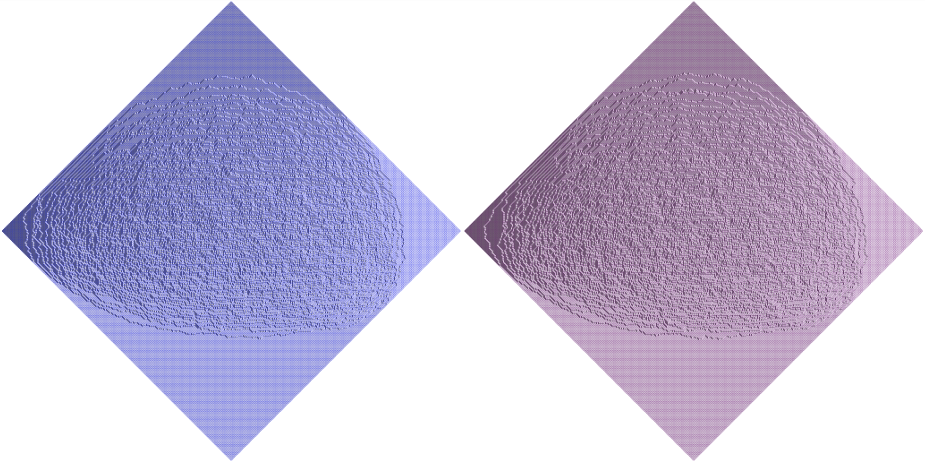
Simulation of a 2-tiling of the rank-256 Aztec diamond at $t = 0.2$.

Simulations



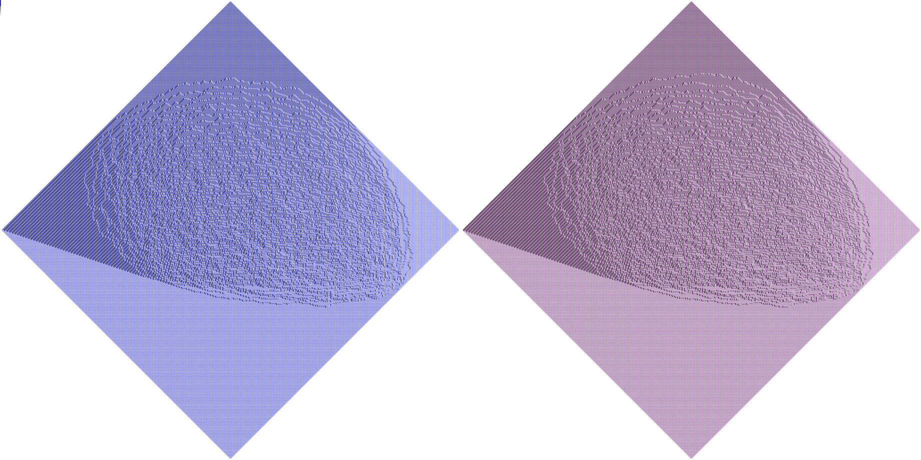
Close-up of southern corner of blue in a 2-tiling of the rank-512 Aztec diamond at $t = 0.2$.

Simulations



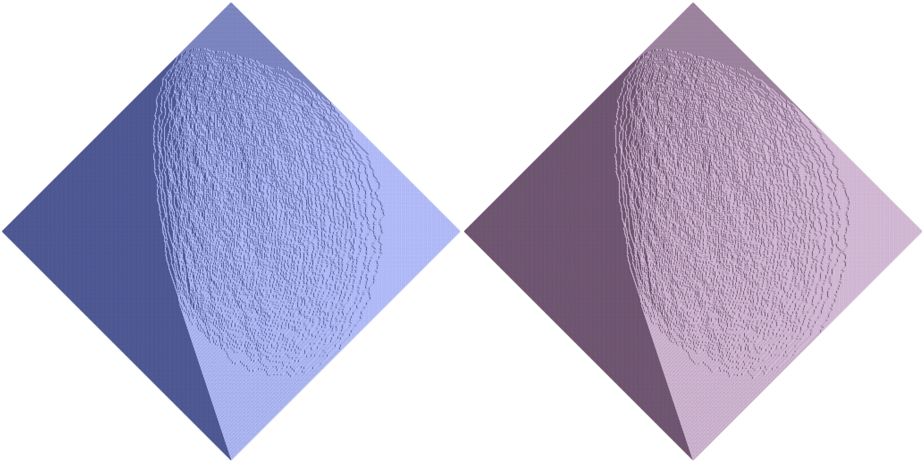
Simulation of a 2-tiling of the rank-256 Aztec diamond at $t = 5$.

Simulations



Simulation of a 2-tiling of the rank-256 Aztec diamond at t very large.

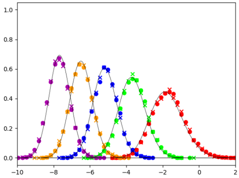
Simulations



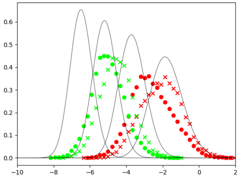
Simulation of a 2-tiling of the rank-256 Aztec diamond at $t = 0$.

Simulations

Fluctuations of the outer-most paths (Courtesy of L. Allen, B. Bertz, H. Kenchareddy through the Madison Experimental Mathematics Lab)

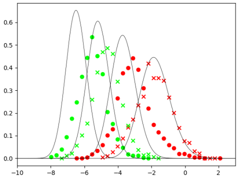


$t = 1$

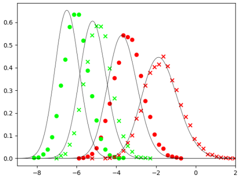


$t = 0.5$

$\times$ = large, $\bullet$ = small



$t = 0.2$



$t = 0$

Remarks

For $t = 0, 1, \infty$ we can prove some things:

- Bijection from $t = 0$ 2-tilings of rank N to normal tilings of rank N.

$$Z_{AD}^{(2)}(X, Y; t) = \prod_{i \le j} (1 + x_i y_j)(1 + x_i y_j t)|_{t=0}$$
$$= \prod_{i \le j} (1 + x_i y_j) = Z_{AD}(X, Y)$$

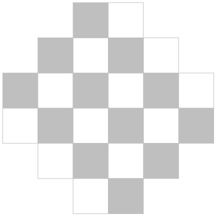
  Can use this to find the arctic curve at $t = 0$, for example.

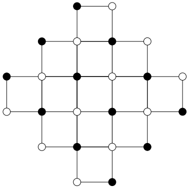
- Symmetry between t and $1/t$. (Reflecting over line $y = x$.)

For generic t, we know very little.

Part 4: Shuffling algorithm

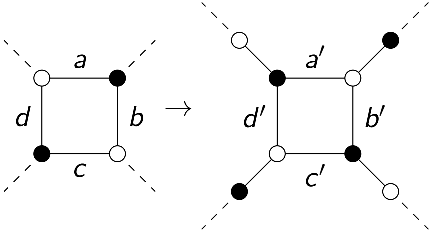
Back to the dimer model





Spider moves

Local move on our graph:



where the weights update as

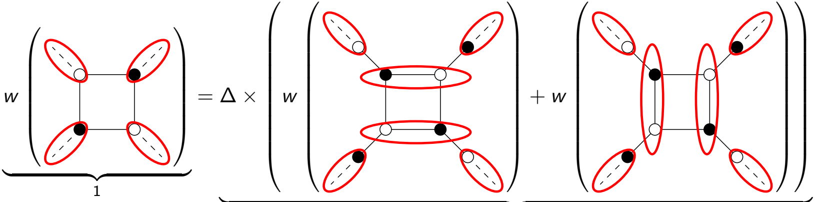
$$a' = \frac{c}{ac+bd}, \quad b' = \frac{d}{ac+bd}, \quad c' = \frac{a}{ac+bd}, \quad d' = \frac{b}{ac+bd}$$

Spider move

Under a spider move the partition function remains unchanged, up to an overall factor,

$$Z = \underbrace{(ac + bd)}_{\Delta} Z'$$

For example:

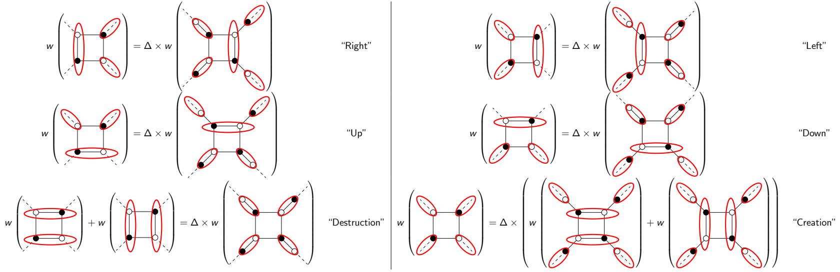


$$(ac+bd)\times(a'c'+b'd')=(ac+bd)\times\frac{ac+bd}{(ac+bd)^2}=1$$

"Creation"

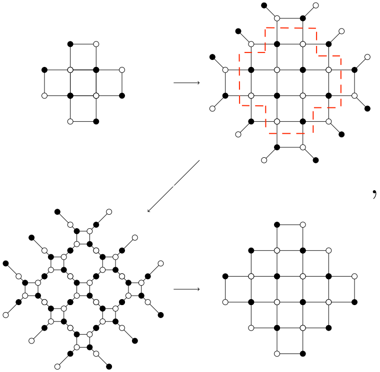
Spider move

Total of six local boundary conditions:



Shuffling

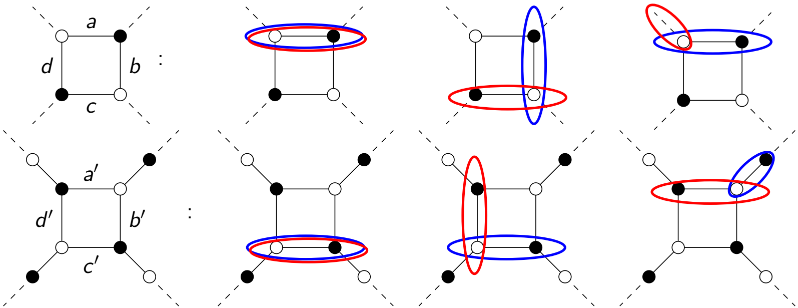
For the Aztec diamond, repeated applications of the spider move allow one to generate large tilings: Embed → spider → contract



$$Z_2 = \left(\prod_{\text{cells } x} \Delta(x) \right) Z_3$$

Spider moves for double dimers

We can generalize the spider move to our interacting double dimers.
Define interactions to be local configurations of the form



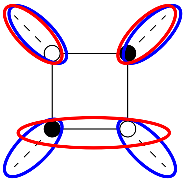
These interactions agree with those of the coupled Aztec diamonds.

Spider moves for double dimers

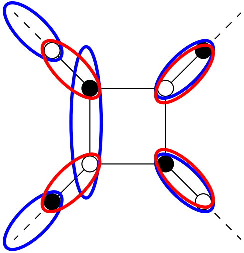
Now there are $6 \times 6 = 36$ possible local boundary conditions which we label by how the dimers 'slide':

$$(\alpha\beta) \in \{c, d, \uparrow, \downarrow, \rightarrow, \leftarrow\}^2$$

$Z_{c\uparrow}$ corresponds to...



$Z'_{\leftarrow d}$ corresponds to...



Spider moves for double dimers

Two important subsets of local boundary conditions:
Define C as the set of boundary conditions $(\alpha\beta)$ for a cell such that

- $\alpha = c$ and $\beta \in \{c, \leftarrow, \downarrow\}$ or
- $\alpha \in \{c, \leftarrow, \downarrow\}$ and $\beta = c$

and define D as the set of boundary conditions $(\alpha\beta)$ such that

- $\alpha = d$ and $\beta \in \{d, \leftarrow, \downarrow\}$ or
- $\alpha \in \{d, \leftarrow, \downarrow\}$ and $\beta = d$.

Spider moves for double dimers

Perform the spider move for both colors. We have

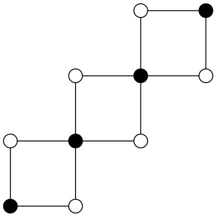
$$Z_{\alpha\beta} = \Delta^2 \Gamma Z'_{\alpha\beta}, \qquad (\alpha\beta) \in C$$
$$Z_{\alpha\beta} = \Delta^2 \Gamma^{-1} Z'_{\alpha\beta} \qquad (\alpha\beta) \in D$$
$$Z_{\alpha\beta} = \Delta^2 Z'_{\alpha\beta} \qquad \text{o.w.}$$

where $\Delta = ac + bd$ and $\Gamma = \frac{ac+bd}{act+bd}$.

- Note in this case the prefactor depends on the the local configuration.
- Can't immediately say that $Z^{(2)}_{N+1} \propto Z^{(2)}_N$.

Generalized shuffling

$$
\begin{aligned}
Z_{\alpha\beta} &= \Delta^2 \Gamma Z'_{\alpha\beta}, \qquad (\alpha\beta) \in C \\
Z_{\alpha\beta} &= \Delta^2 \Gamma^{-1} Z'_{\alpha\beta} \qquad (\alpha\beta) \in D \\
Z_{\alpha\beta} &= \Delta^2 Z'_{\alpha\beta} \qquad \text{o.w.}
\end{aligned}
$$



Lemma (K.-Nicoletti 2023)

For any double dimer configuration on the Aztec diamond of rank N, along each SW-NE diagonal of cells the difference between the number of cells with local boundary condition of type $(\alpha\beta) \in C$ and those of type $(\alpha\beta) \in D$ is equal to 1.

Generalized Shuffling

This implies that if the weights are chosen so that Γ is constant along each SW-NE diagonal then

$$Z_N^{(2)} = \left(\prod_{\text{cells } x} \Delta(x)^2 \right) \left(\prod_{\text{diagonals } d} \Gamma(d) \right) Z_{N+1}^{(2)}$$

Generalized shuffling

Constraint: " if the weights are chosen so that Γ is constant along each SW-NE diagonal"
This is very restrictive.

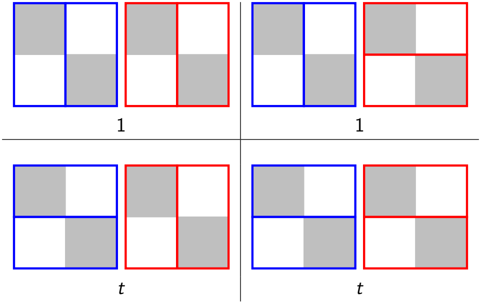
- Since the weights update after each iteration of the shuffling, weights for which the constraint is satisfied for one iteration may not satisfy the constraint for the next iteration.
- Works for uniform weights ($\Gamma = \frac{ac+bd}{act+bd} = \frac{2}{1+t}$ everywhere) since they update to uniform weights.
- Works for "LLT process" weights.
- Doesn't seem to work for 2-periodic weights, for example.

k-tiling shuffling: Step 1

This generalized domino shuffling can be viewed purely in terms of movement of the dominos...

k-tiling shuffling: Step 1

There are 4 rank-1 2-tilings:

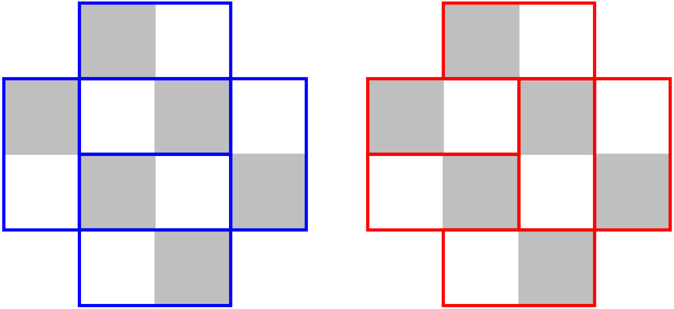


Pick a 2-tiling as follows:

1. With probability $\frac{t}{1+t}$ choose the blue tiling to be horizontal, with probability $\frac{1}{1+t}$ choose vertical.
2. Choose the red tiling to be vertical or horizontal each with probability $\frac{1}{2}$.

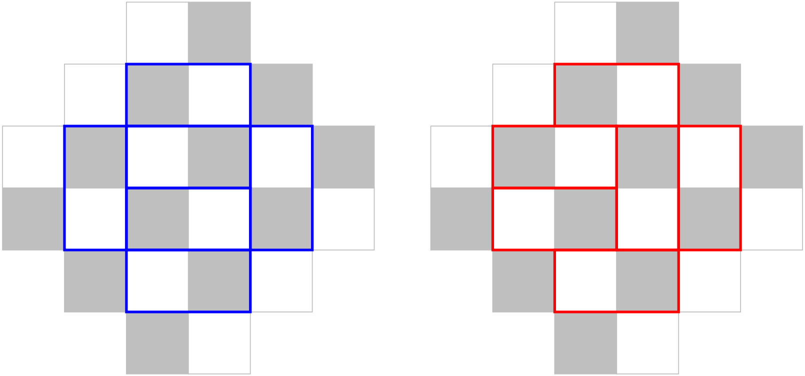
k-tiling shuffling: Step 2

Now suppose we've run the algorithm until we have 2-tiling of rank-k.



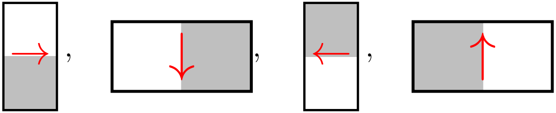
Embed it in an AD of rank-$(k+1)$.

k-tiling shuffling: Step 2

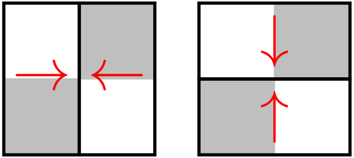


k-tiling shuffling: Step 3

- Slide the dominos one space according to the rules:

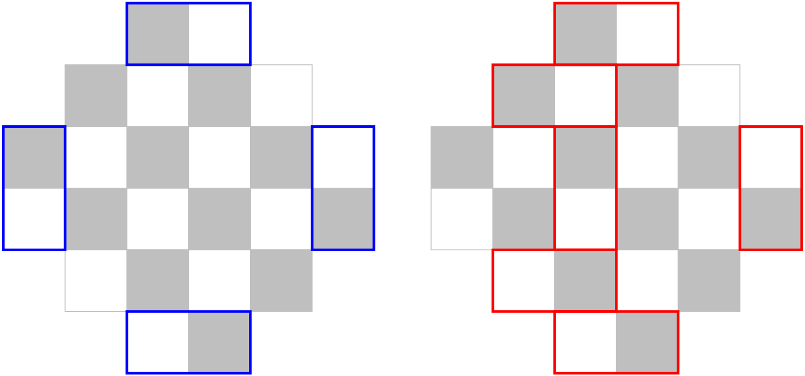


- If two dominos collide, destroy them.



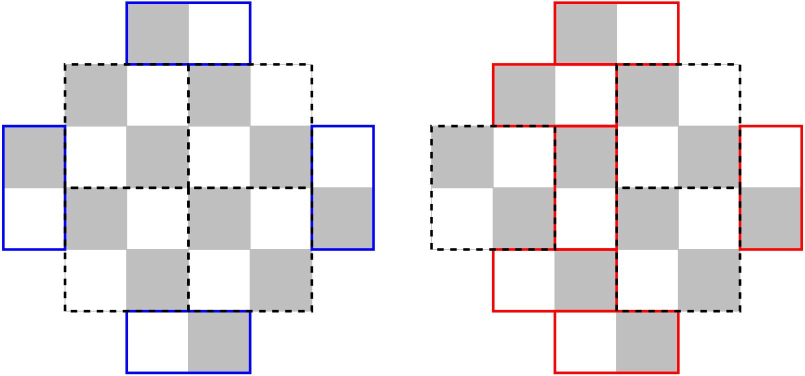
Destruction

k-tiling shuffling: Step 3 cont.



(We swap the checkerboard coloring after to keep with our original convention.)

k-tiling shuffling: Step 4

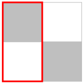
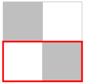


- We are left with a partial tiling of rank-$(k+1)$.
- The empty space in each tiling can be partitioned uniquely into 2×2 squares that all have black square at the top-left.

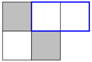
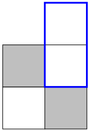
k-tiling shuffling: Step 4 cont.

Fill in the squares according to the rules:

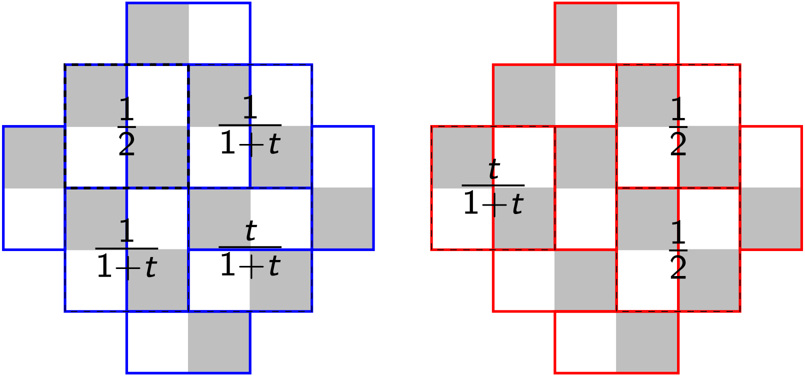
1. First fill in the blue tiling. For each square choose two horizontal dominos with probability $\frac{t^{\#_1}}{1+t^{\#_1}}$ where

$\#_1 = $ 1 if red is  or  or a creation; 0 o.w.

2. Now fill in the red. For each square choose two horizontal dominos with probability $\frac{t^{\#_2}}{1+t^{\#_2}}$ where

$\#_2 = $ 1 if blue is  or ; 0 o.w.

k-tiling shuffling: Step 4 cont.



k-tiling shuffling: Step 5

- Repeat steps 2-4 until you get a tiling of rank-N.

Theorem (K.-Nicoletti 2023)

The probability of getting a 2-tiling $\mathbf{T}_N$ is

$$\mathbb{P}(\mathbf{T}_N) = \frac{w(\mathbf{T}_N)}{Z_{AD}^{(2)}(1, 1; t)}$$

Thank You!

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