Coupled tilings, LLT polynomials, and double dimers

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DIMERS ANR Final Conference
Outline

1. Overview of tilings of the Aztec diamond
2. Defining the coupled tilings (based on work with Sylvie Corteel and Andrew Gitlin: arXiv:2202.06020)
3. Simulations
Part 1: Review of the Aztec diamond
Domino tilings of the Aztec diamond were first introduced by Elkies, Kuperberg, Larsen, and Propp in 1992.

The Aztec diamond of rank $m = 3$ and one possible domino tiling.
There are many ways to view these tilings:

- As a dimer model
- As an example of a Schur process.
- As an integrable vertex model.

For the moment we’ll focus on the last two points.
Domino tilings and sequences of partitions

Assign ‘particles’ and ‘holes’ to our dominos according to the rules.

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Domino tilings and sequences of partitions

\[ \emptyset = \mu^{(1)} \preceq \lambda^{(1)} \preceq' \mu^{(2)} \preceq \ldots \preceq \lambda^{(N-1)} \preceq' \mu^{(N)} \preceq \lambda^{(N)} \preceq' \mu^{(N+1)} = \emptyset \]
Weights

Assign weights to the dominos according to:

- A domino whose left square is on slice $2i - 1$ gets a weight of $x_i$.
- A domino whose right square is on slice $2i - 1$ gets a weight of $y_i$.
- All other dominos get weight of 1.

Then the weight of a tiling

$$\emptyset \preceq \lambda^{(1)} \preceq' \mu^{(2)} \preceq \ldots \preceq \lambda^{(N-1)} \preceq' \mu^{(N)} \preceq \lambda^{(N)} \preceq' \emptyset$$

can be written as

$$s_{\lambda^{(1)}}(x_1)s_{(\lambda^{(1)} / \mu^{(2)})'}(y_1)s_{\lambda^{(2)}}(x_2)\ldots s_{\lambda^{(N)}}(x_N)s_{\lambda^{(N)}'}(y_N)$$
Enumeration

Repeated applications of the Cauchy identity

\[
\sum_{\lambda} s_{\lambda/\nu}(X)s_{\lambda'/\mu'}(Y) = \left( \prod_{i,j} (1 + x_i y_j) \right) \sum_{\lambda} s_{\nu'/\lambda'}(Y)s_{\mu/\lambda}(X).
\]

and branching rule

\[
\sum_{\mu} s_{\lambda/\mu}(X)s_{\mu}(Y) = s_{\lambda}(X, Y)
\]

can be used to show

\[
Z_{AD}(X, Y) = \prod_{i \leq j} (1 + x_i y_j)
\]
Domino tilings as an integrable vertex model

Equivalently, one can view the tilings in terms of integrable vertex models:

There is a weight-preserving bijection between tiling (as a sequence of partitions) and vertex model.
Domino tilings as an integrable vertex model

There is a weight-preserving bijection between tiling (as a sequence of partitions) and vertex model

\[ x_1^2 x_2 x_3 y_2^2 y_3^3 \leftrightarrow (y_1^3 y_2^2 y_3) x_1^2 x_2 x_3 y_2^2 y_3^3 \]

ind. of configuration
Domino tilings as an integrable vertex model

These vertex models satisfy the Yang-Baxter equation:

\[
\sum_{\text{interior paths}} w \left( \begin{array}{c}
J_1 \\
I_1 \\
K_1 \\
J_3 \\
I_3 \\
K_3
\end{array} \right) = \sum_{\text{interior paths}} w \left( \begin{array}{c}
J_1 \\
I_1 \\
K_1 \\
J_3 \\
I_3 \\
K_3
\end{array} \right)
\]

for any fixed choice of boundary condition \( I_1, J_1, K_1, I_3, J_3, K_3 \).
Domino tilings as an integrable vertex model

We can repeatedly apply the YBE to swap rows of the vertex model:

\[
\begin{pmatrix}
\lambda \\
\mu
\end{pmatrix} = \begin{pmatrix}
\lambda \\
\mu
\end{pmatrix}
\]

Then removing the yellow faces (but keeping the weight) gives

\[
w \left( \begin{array}{c}
\lambda \\
\mu
\end{array} \right) = \frac{1}{1 + xy}, \quad w \left( \begin{array}{c}
\lambda \\
\mu
\end{array} \right) = 1
\]
Domino tilings as an integrable vertex model

\[ Z_{AD}(X, Y) = \prod_{i<j} (1 + x_i y_j) \]
Part 2: Coupled tilings of the Aztec diamond
Now rather than a single tiling we will consider a pair of tilings:

We’ll refer to the tilings as being different colors. We order the colors blue < red.
Weights of the coupled tiling

Assign weights to the dominos according to the rules

- A domino of the form \( \begin{array}{c} \ \end{array} \) whose left square is on slice \( 2i - 1 \) gets a weight of \( x_i \).
- A domino of the form \( \begin{array}{c} \ \end{array} \) whose right square is on slice \( 2i - 1 \) gets a weight of \( y_i \).
- All other dominos get a weight of 1.

for each color.

Each ‘interaction’ gives a power of \( t \), \( t \geq 0 \), where we define ‘interaction’ by
Weights of the coupled tiling

In our example,

\[
\begin{pmatrix}
\cell{1}, \cell{2}, \cell{3}, \text{ or } \cell{4}.
\end{pmatrix}
\]

which has weight

\[
\underbrace{x_1^2 x_2 y_2^2 x_3 y_3^2 x_1^3 y_1 y_2 y_3}_{\text{from hor. dominos}} \underbrace{t^4}_{\text{interactions}}.
\]
Where do the weights come from?

If we superimpose the two copies of our five-vertex models, we get a new colored vertex model

\[
\delta_a = \# \text{ colors larger than } a \text{ present}
\]

\[
\gamma_a = \# \text{ colors larger than } a \text{ of the form } a
\]

These vertex models are a degeneration of a vertex model studied by Aggarwal, Borodin, and Wheeler (2021) related to the quantum group \( U_q(\hat{sl}(1|k)) \).
The colored vertex model is still Yang-Baxter integrable \(\text{ (inherited from the vertex model of Aggarwal-Borodin-Wheeler, see also Corteel-Gitlin-K.-Meza 2020)}\)

\[
\begin{align*}
\frac{yxt^\epsilon a}{1+yxt^\epsilon a} & \quad \frac{1}{1+yxt^\epsilon a} & \quad \frac{yxt^\epsilon a}{1+yxt^\epsilon a} & \quad \frac{1}{1+yxt^\epsilon a} & \quad 1 \\
\end{align*}
\]

\[
\epsilon_a = \# \text{ colors larger than a present}
\]

Using the integrability exactly as before, we have

**Theorem (Corteel-Gitlin-K. 2022)**

The partition function for the coupled tilings of the Aztec diamond is given by

\[
Z_{AD}^{(2)}(X, Y; t) = \prod_{i \leq j}(1 + x_i y_j)(1 + x_i y_j t)
\]
Where do the weights come from?

In terms of partitions we now have a bijection between tilings and sequences of 2-tuples of interlacing partitions.

$$\emptyset \preceq \lambda^{(1)} \preceq' \mu^{(2)} \preceq \ldots \preceq \lambda^{(N-1)} \preceq' \mu^{(N)} \preceq \lambda^{(N)} \preceq' \emptyset$$

$$= (\lambda^{(1)}, \lambda^{(1)})$$

The weight of the tiling can be written as

$$t^\# L_{\lambda^{(1)}}(x_1; t) L_{\lambda^{(1)}/\mu^{(2)}}(y_1; t) L_{\lambda^{(2)}/\mu^{(2)}}(x_2; t) L_{\lambda^{(2)}/\mu^{(3)}}(y_2; t) \ldots L_{\lambda^{(N)}/\mu^{(N)}}(x_N; t) L_{\lambda^{(N)}}(y_N; t)$$

The $L$ are called LLT polynomials and are a generalization of the Schur polynomials.
Remarks

- Everything here makes sense for more than 2 colors. Interactions are then counted between every pair of colors.

\[ k \text{ colors: } Z^{(k)}_{AD}(X, Y; t) = \prod_{\ell=0}^{k-1} \prod_{i \leq j} (1 + x_i y_j t^\ell) \]

- Similar constructions can be done for other examples of types of tilings. For example, reverse plane partitions.

\[ Z^{(k)}_{RPP,\lambda}(q; t) = \prod_{\ell=0}^{k-1} \prod_{u \in \lambda} \frac{1}{1 - q^{h(u)} t^\ell} \]
Part 3: Simulations
Simulation of a 2-tiling of the rank-64 Aztec diamond at $t = 1$. 
Simulation of a 2-tiling of the rank-256 Aztec diamond at $t = 1$. 
Simulation of a 2-tiling of the rank-256 Aztec diamond at \( t = 0.2 \).
Close-up of southern corner of blue in a 2-tiling of the rank-512 Aztec diamond at $t = 0.2$. 
Simulation of a 2-tiling of the rank-256 Aztec diamond at $t = 5$. 
Simulation of a 2-tiling of the rank-256 Aztec diamond at $t$ very large.
Simulation of a 2-tiling of the rank-256 Aztec diamond at $t = 0$.
Fluctuations of the outer-most paths (Courtesy of L. Allen, B. Bertz, H. Kenchareddy through the Madison Experimental Mathematics Lab)

$t = 1$

$t = 0.5$

$t = 0.2$

$t = 0$

$\times = \text{large, } \bullet = \text{small}$
For $t = 0, 1, \infty$ we can prove some things:

- Bijection from $t = 0$ 2-tilings of rank $N$ to normal tilings of rank $N$.

$$Z_{AD}^{(2)}(X, Y; t) = \prod_{i \leq j} (1 + x_i y_j)(1 + x_i y_j t)|_{t=0}$$

$$= \prod_{i \leq j} (1 + x_i y_j) = Z_{AD}(X, Y)$$

Can use this to find the arctic curve at $t = 0$, for example.

- Symmetry between $t$ and $1/t$. (Reflecting over line $y = x$.)

For generic $t$, we know very little.
Part 4: Shuffling algorithm
Back to the dimer model
Spider moves

Local move on our graph:

\[
\begin{align*}
    a' &= \frac{c}{ac + bd}, \quad b' = \frac{d}{ac + bd}, \quad c' = \frac{a}{ac + bd}, \quad d' = \frac{b}{ac + bd}
\end{align*}
\]
Under a spider move the partition function remains unchanged, up to an overall factor,

\[ Z = (ac + bd) Z' \]

For example:

\[
\begin{align*}
    w & = \Delta \times w \\
    1 & = (ac + bd) \times (ac + bd) = 1
\end{align*}
\]
Spider move

Total of six local boundary conditions:

\[w \left( \begin{array}{c} \text{Up} \\ \end{array} \right) = \Delta \times w \left( \begin{array}{c} \text{Up} \\ \end{array} \right)\]

\[w \left( \begin{array}{c} \text{Right} \\ \end{array} \right) = \Delta \times w \left( \begin{array}{c} \text{Right} \\ \end{array} \right)\]

\[w \left( \begin{array}{c} \text{Down} \\ \end{array} \right) = \Delta \times w \left( \begin{array}{c} \text{Down} \\ \end{array} \right)\]

\[w \left( \begin{array}{c} \text{Destruction} \\ \end{array} \right) = \Delta \times w \left( \begin{array}{c} \text{Destruction} \\ \end{array} \right)\]

\[w \left( \begin{array}{c} \text{Creation} \\ \end{array} \right) = \Delta \times w \left( \begin{array}{c} \text{Creation} \\ \end{array} \right)\]
Shuffling

For the Aztec diamond, repeated applications of the spider move allow one to generate large tilings: Embed $\rightarrow$ spider $\rightarrow$ contract

$$Z_2 = \left( \prod_{\text{cells } x} \Delta(x) \right) Z_3$$
We can generalize the spider move to our interacting double dimers. Define interactions to be local configurations of the form:

These interactions agree with those of the coupled Aztec diamonds.
Now there are $6 \times 6 = 36$ possible local boundary conditions which we label by how the dimers ‘slide’:

$$(\alpha \beta) \in \{c, d, \uparrow, \downarrow, \rightarrow, \leftarrow\}^2$$

$Z_{c\uparrow}$ corresponds to... $Z'_{\leftarrow d}$ corresponds to...
Two important subsets of local boundary conditions:
Define $C$ as the set of boundary conditions $(\alpha \beta)$ for a cell such that
- $\alpha = c$ and $\beta \in \{c, \leftarrow, \downarrow\}$ or
- $\alpha \in \{c, \leftarrow, \downarrow\}$ and $\beta = c$
and define $D$ as the set of boundary conditions $(\alpha \beta)$ such that
- $\alpha = d$ and $\beta \in \{d, \leftarrow, \downarrow\}$ or
- $\alpha \in \{d, \leftarrow, \downarrow\}$ and $\beta = d$. 
Spider moves for double dimers

Perform the spider move for both colors. We have

\[ Z_{\alpha\beta} = \Delta^2 \Gamma Z'_{\alpha\beta}, \quad (\alpha\beta) \in C \]
\[ Z_{\alpha\beta} = \Delta^2 \Gamma^{-1} Z'_{\alpha\beta}, \quad (\alpha\beta) \in D \]
\[ Z_{\alpha\beta} = \Delta^2 Z'_{\alpha\beta} \quad \text{o.w.} \]

where \( \Delta = ac + bd \) and \( \Gamma = \frac{ac + bd}{act + bd} \).

- Note in this case the prefactor depends on the the local configuration.
- Can’t immediately say that \( Z_{N+1}^{(2)} \propto Z_N^{(2)} \).
Lemma (K.-Nicoletti 2023)

For any double dimer configuration on the Aztec diamond of rank $N$, along each SW-NE diagonal of cells the difference between the number of cells with local boundary condition of type $(\alpha \beta) \in C$ and those of type $(\alpha \beta) \in D$ is equal to 1.
Generalized Shuffling

This implies that if the weights are chosen so that $\Gamma$ is constant along each SW-NE diagonal then

$$Z_N^{(2)} = \left( \prod_{\text{cells } x} \Delta(x)^2 \right) \left( \prod_{\text{diagonals } d} \Gamma(d) \right) Z_{N+1}^{(2)}$$
Generalized shuffling

Constraint: “if the weights are chosen so that $\Gamma$ is constant along each SW-NE diagonal”
This is very restrictive.

- Since the weights update after each iteration of the shuffling, weights for which the constraint is satisfied for one iteration may not satisfy the constraint for the next iteration.

- Works for uniform weights ($\Gamma = \frac{ac+bd}{act+bd} = \frac{2}{1+t}$ everywhere) since they update to uniform weights.

- Works for “LLT process” weights.

- Doesn’t seem to work for 2-periodic weights, for example.
This generalized domino shuffling can be viewed purely in terms of movement of the dominos...
$k$-tiling shuffling: Step 1

There are 4 rank-1 2-tilings:

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$t$

Pick a 2-tiling as follows:

1. With probability $\frac{t}{1+t}$ choose the blue tiling to be horizontal, with probability $\frac{1}{1+t}$ choose vertical.
2. Choose the red tiling to be vertical or horizontal each with probability $\frac{1}{2}$. 
Now suppose we’ve run the algorithm until we have 2-tiling of rank-\(k\).

Embed it in an AD of rank-\((k + 1)\).
$k$-tiling shuffling: Step 2
- Slide the dominos one space according to the rules:

- If two dominos collide, destroy them.
(We swap the checkerboard coloring after to keep with our original convention.)
- We are left with a partial tiling of rank-$(k + 1)$.
- The empty space in each tiling can be partitioned uniquely into $2 \times 2$ squares that all have black square at the top-left.
Fill in the squares according to the rules:

1. First fill in the blue tiling. For each square choose two horizontal dominos with probability \( \frac{t \#_1}{1 + t \#_1} \) where

\[
\#_1 = \begin{cases} 
1 & \text{if red is } \ \\
0 & \text{otherwise (o.w.)}
\end{cases}
\]

2. Now fill in the red. For each square choose two horizontal dominos with probability \( \frac{t \#_2}{1 + t \#_2} \) where

\[
\#_2 = \begin{cases} 
1 & \text{if blue is } \ \\
0 & \text{otherwise (o.w.)}
\end{cases}
\]
$k$-tiling shuffling: Step 4 cont.
**k-tiling shuffling: Step 5**

- Repeat steps 2-4 until you get a tiling of rank-$N$.

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**Theorem (K.-Nicoletti 2023)**

*The probability of getting a 2-tiling $T_N$ is*

$$
P(T_N) = \frac{w(T_N)}{Z_{AD}^{(2)}(1, 1; t)}$$
Thank You!


