Aalto University
School of Science
and Technology

FINNISH CENTRE OF EXCELLENCE
IN RANDOMNESS AND
STRUCTURES 2022–2029

Local fields of the discrete GFF and the free boson CFT

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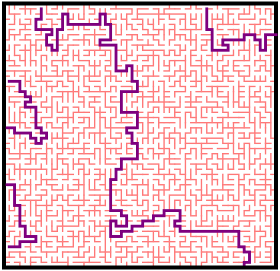
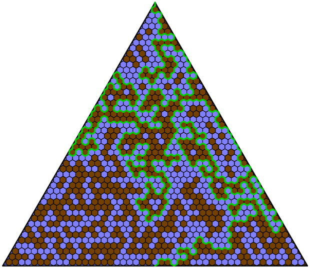
1. Introduction

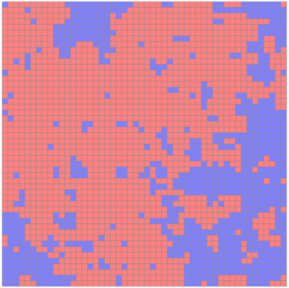
Intro: Two-dimensional statistical physics



uniform spanning tree



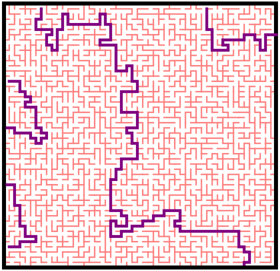
percolation



Ising model

. . . etc. . . .

Intro: Two-dimensional statistical physics



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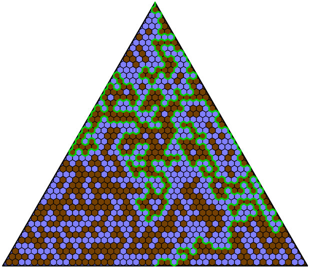

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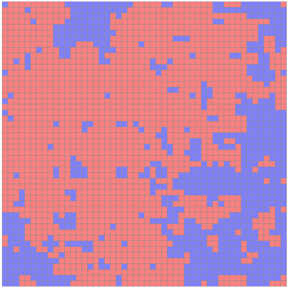

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. . . etc. . . .

Conventional wisdom: *“Any interesting scaling limit of any two-dimensional random lattice model is described by a Conformal Field Theory (CFT).“*

Intro: Two-dimensional statistical physics

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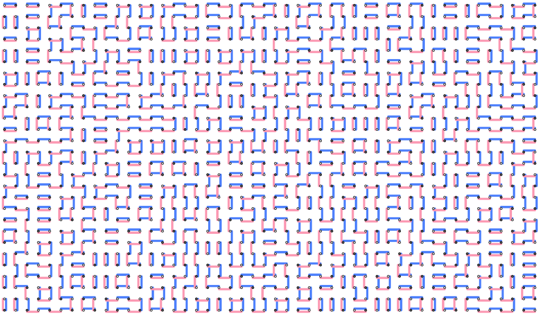

double dimer model!

...etc. ...

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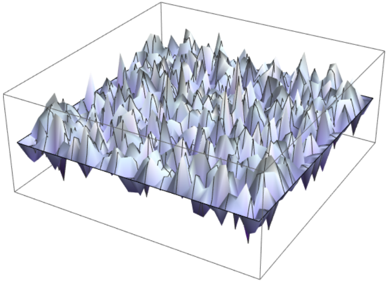
Intro: Two-dimensional statistical physics



discrete Gaussian Free Field

TODAY

Conventional wisdom: *“Any interesting scaling limit of any two-dimensional random lattice model is described by a Conformal Field Theory (CFT).“*

Intro: CFT scaling limit statement — a proposal

Local fields of a CFT

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Local fields of a lattice model

Local fields of a CFT

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Bijective correspondence of "local fields"

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$\circlearrowleft$

$2 \times$ Virasoro algebra

Intro: CFT scaling limit statement — a proposal

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Expected values in $\Omega_\delta^\# \subset \delta\mathbb{Z}^2$

$$\mathsf{E}_{\Omega_\delta^\#}\Big[\prod_{j=1}^n F_j(z_j^\#)\Big]$$

CFT correlation functions in $\Omega \subset \mathbb{C}$

$$\langle F_1(z_1)\cdots F_n(z_n)\rangle_\Omega$$

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$$\mathsf{E}_{\Omega_\delta^\#}\Big[\prod_{j=1}^n \Big(\frac{F_j(z_j^\#)}{\delta^{D(F_j)}}\Big)\Big] \underset{\delta\to 0}{\longrightarrow} \langle F_1(z_1)\cdots F_n(z_n)\rangle_\Omega$$

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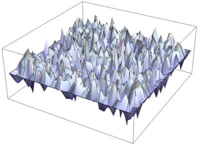
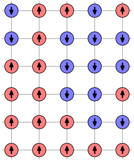
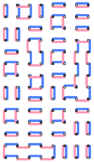
$$\langle F_1(z_1)\cdots F_n(z_n)\rangle_\Omega$$

and correlations converge in the scaling limit,
with renormalization of fields F_j determined by their scaling dimensions $D(F_j)$ defined as the $\mathsf{L}_0 + \bar{\mathsf{L}}_0$ eigenvalues.

Intro: Current status of the proposal

	discrete GFF	**Ising model**	**double dimer /** symplectic fermion	?
Virasoro $\curvearrowright$ local fields	$c = 1$ ✓ [HKV22]	$c = \frac{1}{2}$ ✓ [HKV22]	$c = -2$ ✓ [A23]	?
1-to-1 correspondece	✓ [ABK23+]	?	?	?
scaling limit correlations	✓ [ABK23+]	?	?	?

[HKV22] Hongler & K & Viklund: `https://arxiv.org/abs/1307.4104`

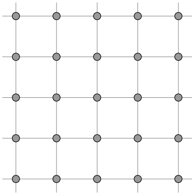
[A23] Adame-Carrillo: `https://arxiv.org/abs/2304.08163`

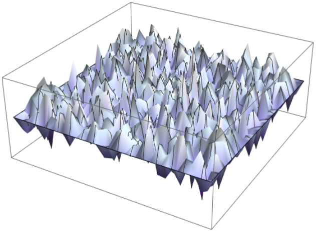
[ABK23+] Adame-Carrillo & Behzad & K: in preparation

II. Lattice local fields of the discrete Gaussian Free Field (DGFF)

Discrete Gaussian Free Field

- $\Omega \subsetneq \mathbb{C}$ open, simply connected
- lattice approximation: $\Omega_\delta^\# \subset \mathbb{C}_\delta := \delta\mathbb{Z}^2$





Discrete Gaussian Free Field

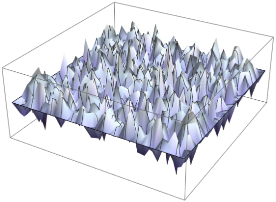
$$\Phi = \big(\Phi(z)\big)_{z \in \Omega_\delta^\#}$$

centered Gaussian field on vertices of discrete domain $\Omega_\delta^\#$

covariance
$$\mathsf{E}_{\Omega_\delta^\#}[\Phi(z)\Phi(w)] = 4\pi\, \mathsf{G}_{\Omega_\delta^\#}(z,w)$$

discrete Green's function
$$\begin{cases} \triangle_\delta \mathsf{G}_{\Omega_\delta^\#}(\cdot, w) = -\delta_w(\cdot) \\ \mathsf{G}_{\Omega_\delta^\#}(\cdot, w) = 0 \ \text{ on } \partial\Omega_\delta^\# \end{cases}$$

Local fields of the DGFF



$$\Phi = \big(\Phi(z)\big)_{z \in \Omega_\delta^\#} \qquad \text{DGFF}$$

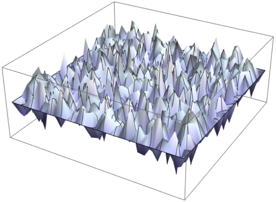
Examples of “local fields”:

* $F(z) = \Phi(z)$
* $F(z) = \frac{1}{2}\Phi(z+\delta) - \frac{1}{2}\Phi(z-\delta)$
* $F(z) = 361\,\big(\Phi(z - \delta + 2\delta\mathbb{i})\big)^{37}$

Local fields of the DGFF

Space of local fields

$\mathcal{P} = \mathbb{C}[\varphi(x) : x \in \mathbb{Z}^2]$



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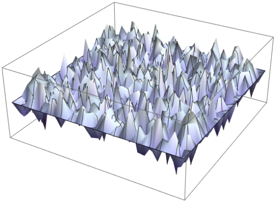

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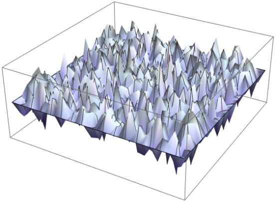
Null fields "zero inside correlations"

- $F \in \mathcal{P}$ null if $\exists R > 0$ s.t.
  $\mathsf{E}_{\Omega_\delta^\#}\Big[F(z)\ \prod_{j=1}^n \Phi(w_j)\Big] = 0$
  whenever $\|z - w_j\|_1 > R\delta \ \ \forall j$
  and $\min_{w\in\mathbb{C}_\delta\setminus\Omega_\delta^\#} \|z-w\|_1 > R\delta$

$\rightsquigarrow$ $\mathcal{N} \subset \mathcal{P}$ (sub)space of null fields



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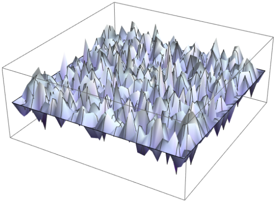

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Example of "null field":

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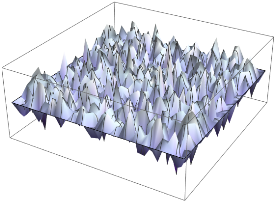

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$\mathcal{P}/\mathcal{N}$ — equivalence classes of local fields, "same correlations"

Main result A

Theorem A [Hongler & K. & Viklund, 2017]

The space $\mathcal{P}/\mathcal{N}$ of correlation equivalence classes of local fields of the discrete Gaussian free field forms a representation of the Virasoro algebra with central charge $c = 1$.

II. The Fock space of local fields of the Gaussian Free Field

The (chiral) Fock representation abstractly

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$\mathfrak{H}$ **Heisenberg algebra** ("chiral symmetry alg. of free boson")

generators $\mathsf{J}_\ell \in \mathfrak{H}$ for $\ell \in \mathbb{Z}$

relations $[\mathsf{J}_\ell, \mathsf{J}_m] := \mathsf{J}_\ell \mathsf{J}_m - \mathsf{J}_m \mathsf{J}_\ell = \ell \, \delta_{\ell+m,0} \, 1$

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$\mathscr{F} := \mathfrak{H} / (\mathsf{J}_\ell : \ell \geq 0)$ (chargeless) **Fock representation** of $\mathfrak{H}$

(quotient of $\mathfrak{H}$ by the left ideal generated by J_ℓ, $\ell \geq 0$)

basis: $\mathsf{J}_{-k_m} \cdots \mathsf{J}_{-k_2} \mathsf{J}_{-k_1} [1] \in \mathscr{F}$ with $0 < k_1 \leq k_2 \leq \cdots \leq k_m$

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Remark: For any representation V of $\mathfrak{H}$ and vector $v \in V \setminus \{0\}$ satisfying $\mathsf{J}_\ell v = 0$ for $\ell \geq 0$, we have $\mathfrak{H} v \cong \mathscr{F}$.

(Verma-like universal property + irreducibility of $\mathscr{F}$)

Bosonic Sugawara construction

commutator $[A, B] \;:=\; A \circ B - B \circ A$

Lemma (bosonic Sugawara construction)

[well-known]

Suppose:

- V a representation of $\mathfrak{H}$
- $\forall v \in V \quad \exists N \in \mathbb{Z} \;:\; \ell \geq N \;\Rightarrow\; \mathrm{J}_\ell \, v = 0$

Define:

$$L_n \;:=\; \frac{1}{2} \sum_{\ell<0} \mathrm{J}_\ell \, \mathrm{J}_{n-\ell} + \frac{1}{2} \sum_{\ell \geq 0} \mathrm{J}_{n-\ell} \, \mathrm{J}_\ell \quad \text{for } n \in \mathbb{Z}$$

Then:

- $L_n \colon V \to V$ is well-defined
- $[L_n, L_m] = (n-m)\, L_{n+m} + \frac{n^3-n}{12}\, \delta_{n+m,0}\, \mathrm{id}_V$

$\therefore$ V Virasoro representation, central charge $c = 1$

Corollary (Virasoro action on the Fock representation)

$\mathscr{F}$ carries a representation of Virasoro algebra with $c = 1$.

The full Fock space — two chiralities

A full CFT has two commuting chiral algebras: *"holomorphic"* and *"antiholomorphic"*. Distinguish the latter by a bar on top.

- two-chiral Heisenberg algebra $\mathfrak{H} \otimes \overline{\mathfrak{H}}$ (with $\overline{\mathfrak{H}} \cong \mathfrak{H}$)
- **Fock space** $\mathscr{F} \otimes \overline{\mathscr{F}}$ ($\mathfrak{H}$ acts on 1st factor, $\overline{\mathfrak{H}}$ on 2nd factor)

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Remark 1: $\mathscr{F} \otimes \overline{\mathscr{F}}$ has two commuting Virasoro actions ($c = 1$), generators L_n, $\bar{\mathsf{L}}_n$ for $n \in \mathbb{Z}$. (Sugawara with $\mathfrak{H}$ and $\overline{\mathfrak{H}}$)

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Remark 1: $\mathscr{F} \otimes \overline{\mathscr{F}}$ has two commuting Virasoro actions ($c = 1$), generators L_n, $\bar{\mathsf{L}}_n$ for $n \in \mathbb{Z}$. (Sugawara with $\mathfrak{H}$ and $\overline{\mathfrak{H}}$)

Remark 2: For a basis vector $F = \mathsf{J}_{-k_m} \cdots \mathsf{J}_{-k_1} \bar{\mathsf{J}}_{-k'_{m'}} \cdots \bar{\mathsf{J}}_{-k'_1} \mathbb{I}$,

$$\mathsf{L}_0 F = \Big(\sum_{j=1}^{m} k_j \Big) F, \qquad \bar{\mathsf{L}}_0 F = \Big(\sum_{j=1}^{m'} k'_j \Big) F.$$

The full Fock space — two chiralities

A full CFT has two commuting chiral algebras: *“holomorphic”* and *“antiholomorphic”*. Distinguish the latter by a bar on top.

- two-chiral Heisenberg algebra $\mathfrak{H} \otimes \overline{\mathfrak{H}}$ (with $\overline{\mathfrak{H}} \cong \mathfrak{H}$)
- **Fock space** $\mathscr{F} \otimes \overline{\mathscr{F}}$ ($\mathfrak{H}$ acts on 1st factor, $\overline{\mathfrak{H}}$ on 2nd factor)

  “vacuum” / “identity”: $\mathbb{I} = [1] \otimes [1] \in \mathscr{F} \otimes \overline{\mathscr{F}}$

  basis: $\mathsf{J}_{-k_m} \cdots \mathsf{J}_{-k_1} \bar{\mathsf{J}}_{-k'_{m'}} \cdots \bar{\mathsf{J}}_{-k'_1} \mathbb{I} \in \mathscr{F} \otimes \overline{\mathscr{F}}$

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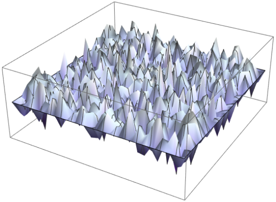
Remark 3: For any repr. V of $\mathfrak{H} \otimes \overline{\mathfrak{H}}$ and vector $v \in V \setminus \{0\}$ satisfying $\mathsf{J}_\ell v = 0 = \bar{\mathsf{J}}_\ell v$ for $\ell \geq 0$, we have $(\mathfrak{H} \otimes \overline{\mathfrak{H}}) v \cong \mathscr{F} \otimes \overline{\mathscr{F}}$.

The Gaussian Free Field (GFF)

GFF on $\Omega \subsetneq \mathbb{C}$:



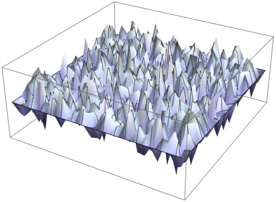
... morally a Gaussian "$(\phi(z))_{z\in\Omega}$" with density

$$\propto \exp\Big(-\frac{1}{8\pi} \iint_\Omega \|\nabla\phi(z)\|^2 \, \mathrm{d}^2 z \Big)$$

The Gaussian Free Field (GFF)

GFF on $\Omega \subsetneq \mathbb{C}$: a random distribution $\phi \in \mathcal{D}'(\Omega)$ with centered Gaussian law and covariance (at $f_1, f_2 \in C_c^\infty(\Omega)$)

$$\mathrm{E}[\langle \phi, f_1 \rangle \langle \phi, f_2 \rangle] = \iint_\Omega \mathrm{d}^2 z \iint_\Omega \mathrm{d}^2 w \; f_1(z) \, \mathrm{C}_\Omega(z,w) \, f_2(w)$$



. . . morally a Gaussian "$(\phi(z))_{z\in\Omega}$" with density

$$\propto \exp\Big(- \frac{1}{8\pi} \iint_\Omega \|\nabla \phi(z)\|^2 \, \mathrm{d}^2 z \Big)$$

covariance kernel $\mathrm{C}_\Omega(z,w) = 4\pi \, \mathrm{G}_\Omega(z,w)$ with

(Dirichlet) **Green's function**

$$\begin{cases} \triangle \mathrm{G}_\Omega(\cdot, w) \equiv 0 \quad \text{on } \Omega \setminus \{w\} \\ \mathrm{G}_\Omega(z,w) = \frac{-1}{2\pi} \log|z-w| + \mathcal{O}(1) \\ \mathrm{G}_\Omega(z,w) \to 0 \;\; \text{as } z \to \partial\Omega \end{cases}$$

Basic correlation functions of the GFF

$$\mathsf{E}\Big[\prod_{j=1}^{n} \langle \phi, f_j \rangle\Big] = \sum_{P} \prod_{\{i,j\} \in P} \mathsf{E}[\langle \phi, f_i \rangle \langle \phi, f_j \rangle] \qquad \text{(Wick's formula)}$$

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$$= \int\!\cdots\!\int_{\Omega\times\cdots\times\Omega} \mathrm{d}^2 z_1 \cdots \mathrm{d}^2 z_n \; f_1(z_1)\cdots f_n(z_n) \underbrace{\Big(\sum_{P}\prod_{\{i,j\}\in P} \mathsf{C}_\Omega(z_i, z_j)\Big)}_{=\langle\phi(z_1)\cdots\phi(z_n)\rangle_\Omega}$$

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For the simplest possible CFT, we consider "GFF up to additive constant", i.e., the derivatives of GFF are considered local fields, but GFF itself is not.

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Currents $\mathrm{J}=\mathbb{i}\partial\phi$, $\bar{\mathrm{J}}=-\mathbb{i}\bar{\partial}\phi$ with correlation functions

$$\Big\langle\prod_{i=1}^{n}\mathrm{J}(z_i)\prod_{\ell=1}^{m}\bar{\mathrm{J}}(w_\ell)\Big\rangle_\Omega := \mathbb{i}^n(-\mathbb{i})^m\frac{\partial^{n+m}}{\partial z_1\cdots\partial\bar{w}_m}\Big\langle\prod_{i=1}^{n}\phi(z_i)\prod_{\ell=1}^{m}\phi(w_\ell)\Big\rangle_\Omega$$

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$$\sim\frac{1}{(z_j-z_k)^2}\Big\langle\prod_{i\notin\{j,k\}}\mathrm{J}(z_i)\,\prod_{\ell}\bar{\mathrm{J}}(w_\ell)\Big\rangle_\Omega + \mathcal{O}(1) \qquad \sim\frac{1}{(\bar{w}_r-\bar{w}_s)^2}\Big\langle\prod_{i}\mathrm{J}(z_i)\,\prod_{\ell\notin\{r,s\}}\bar{\mathrm{J}}(w_\ell)\Big\rangle_\Omega + \mathcal{O}(1)$$

General correlation functions by recursive OPEs

Recursively define $$\left\langle \prod_i \mathsf{J}(\zeta_i) \prod_{i'} \bar{\mathsf{J}}(\xi_{i'}) \prod_j F_j(z_j) \right\rangle_\Omega$$

for symbols of the form $F_j = \mathsf{J}_{k_m} \cdots \mathsf{J}_{k_1} \bar{\mathsf{J}}_{k'_{m'}} \cdots \bar{\mathsf{J}}_{k'_1} \mathbb{I}$

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$$\left\langle \mathrm{J}(\zeta) F_j(z) \cdots \right\rangle_\Omega = \sum_{k=-\infty}^{K} (\zeta - z)^{-1-k} \left\langle (\mathrm{J}_k F_j)(z) \cdots \right\rangle_\Omega$$

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General correlation functions by recursive OPEs

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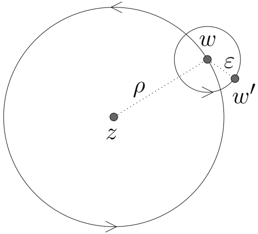
- to replace F_j by $\bar{\mathrm{J}}_k F_j$, $k \in \mathbb{Z}$, use OPE with $\bar{\mathrm{J}}$ similarly

Correlation functions of Fock space local fields



Multilinearly extend from the symbols:

$$(F_1, \ldots, F_n) \mapsto \langle F_1(z_1) \cdots F_n(z_n) \rangle_\Omega$$

The extension respects, in each entry, the defining relations of $\mathfrak{H} \otimes \overline{\mathfrak{H}}$ and $\mathscr{F} \otimes \overline{\mathscr{F}}$, so it factors through the tensor product of Fock spaces and defines:

$$(\mathscr{F} \otimes \overline{\mathscr{F}})^{\otimes n} \to \mathcal{C}^\omega\big(\{(z_1, \ldots, z_n) \in \Omega^n \;\big|\; z_i \neq z_j \text{ for } i \neq j\}\big)$$

CFT of the gradient of GFF:

- space of local fields $\mathscr{F} \otimes \overline{\mathscr{F}}$.
- $F_1, \ldots, F_n \in \mathscr{F} \otimes \overline{\mathscr{F}}$, $\quad \Omega \subset \mathbb{C}$, $\quad$ distinct $z_1, \ldots, z_n \in \Omega$
  $\rightsquigarrow$ correlation function $\langle F_1(z_1) \cdots F_n(z_n) \rangle_\Omega$

IV. Main results

Main results

Theorem A [Hongler & K. & Viklund, 2017]

The space $\mathcal{P}/\mathcal{N}$ of correlation equivalence classes of local fields of the discrete Gaussian free field forms a representation of the Virasoro algebra with central charge $c = 1$ (and also a representation of the Heisenberg algebra).

Main results

Theorem A

[Hongler & K. & Viklund, 2017]

The space $\mathcal{P}/\mathcal{N}$ of correlation equivalence classes of local fields of the discrete Gaussian free field forms a representation of the Virasoro algebra with central charge $c = 1$ (and also a representation of the Heisenberg algebra).

Let $\mathcal{P}_\nabla = \mathbb{C}[\varphi(z) - \varphi(0) : z \in \mathbb{Z}^2] \subset \mathcal{P}$ "polynomials in the gradient"

Theorem B

[Adame-Carrillo & Behzad & K., 2023$^+$]

$$\mathcal{P}_\nabla/\mathcal{N} \cong \mathscr{F} \otimes \overline{\mathscr{F}}$$

as a representation of two commuting Virasoro algebras (and also of two commuting Heisenberg algebras).

Main results

Assuming:

- $F_1, \ldots, F_n \in \mathcal{P}_\nabla / \mathcal{N} \cong \mathscr{F} \otimes \overline{\mathscr{F}}$
  $\mathrm{L}_0 F_j = \Delta_j F_j$ and $\bar{\mathrm{L}}_0 F_j = \bar{\Delta}_j F_j$
- $\Omega_\delta^\# \to \Omega \subsetneq \mathbb{C}$ (open) (in Carathéodory sense)
- $z_1, \ldots, z_n \in \Omega$ distinct, $z_j^\# \in \Omega_\delta^\#$ closest to z_j

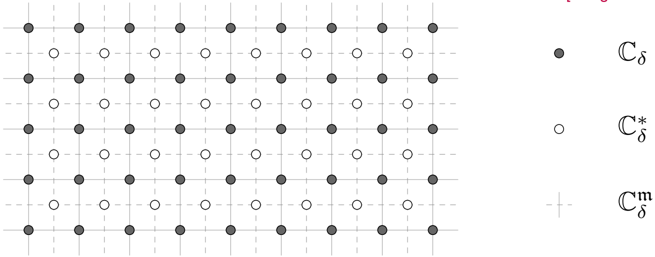
Theorem C [Adame-Carrillo & Behzad & K., 2023$^+$]

$$\mathsf{E}_{\Omega_\delta^\#}\Big[\prod_{j=1}^n \Big(\frac{F_j(z_j^\#)}{\delta^{\Delta_j + \bar{\Delta}_j}}\Big)\Big] \underset{\delta \to 0}{\longrightarrow} \langle F_1(z_1) \cdots F_n(z_n) \rangle_\Omega .$$

V. Ingredients of proofs

Ingredients of the proof of Theorem A

[Hongler & K. & Viklund]



- Discrete currents $\mathrm{J}(z) = \mathbb{i}\, \partial^\bullet_\delta \Phi(z), \quad \bar{\mathrm{J}}(z) = -\mathbb{i}\, \bar{\partial}^\bullet_\delta \Phi(z)$.
- Discrete contour integration $\oint_\#$ (with good properties).
- Discrete holomorphic monomial functions (residue calc.).
  - ⇝ Can define currents' Laurent modes as operators on $\mathcal{P}/\mathcal{N}$.
- Heisenberg commutation relations for current modes.
- Sugawara construction of Virasoro from the current modes.

Ingredients of the proof of Theorem A

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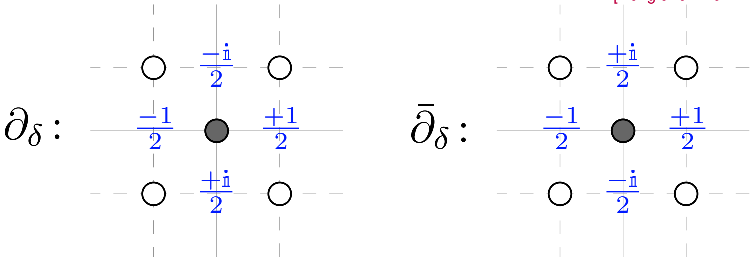

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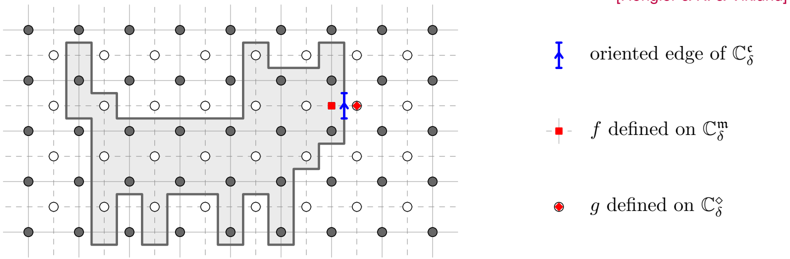

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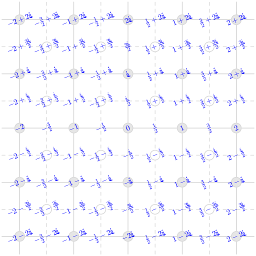

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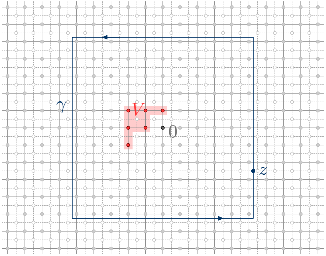

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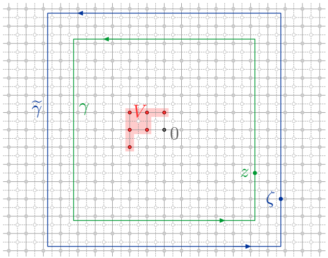

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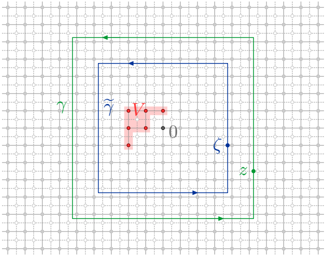

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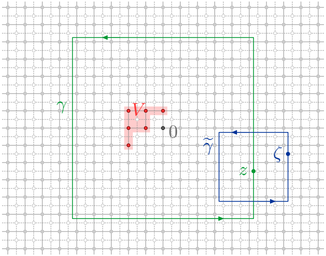

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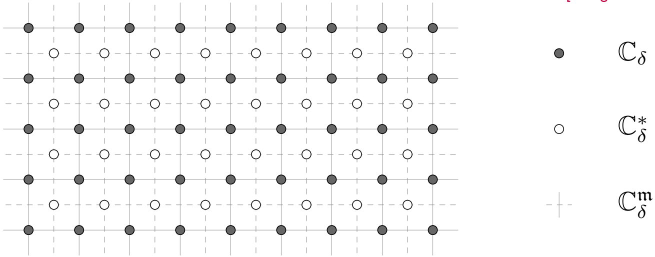

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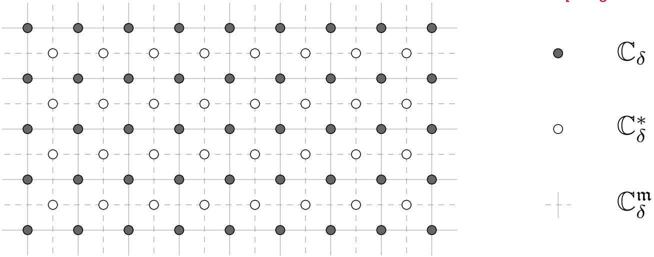
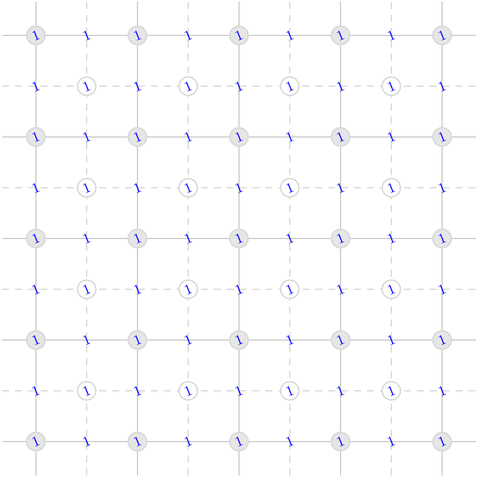
Ingredients of the proof of Theorem A

[Hongler & K. & Viklund]



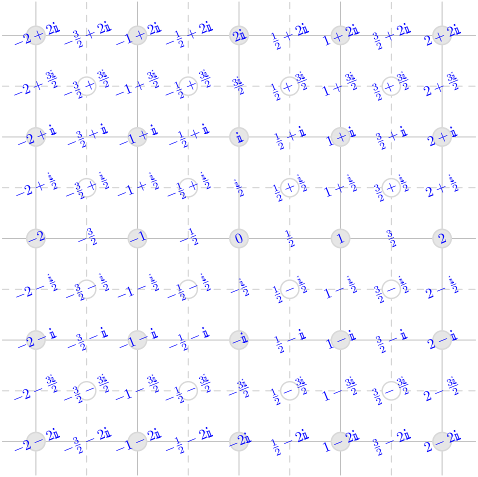
- Discrete currents $\mathrm{J}(z) = \mathbb{i}\,\partial_\delta^\bullet\,\Phi(z)$, $\bar{\mathrm{J}}(z) = -\mathbb{i}\,\bar{\partial}_\delta^\bullet\,\Phi(z)$.
- Discrete contour integration $\oint_\#$ (with good properties).
- Discrete holomorphic monomial functions (residue calc.).
  - ⇝ Can define currents' Laurent modes as operators on $\mathcal{P}/\mathcal{N}$.
- Heisenberg commutation relations for current modes.
- Sugawara construction of Virasoro from the current modes.

Discrete monomial functions (example 0)



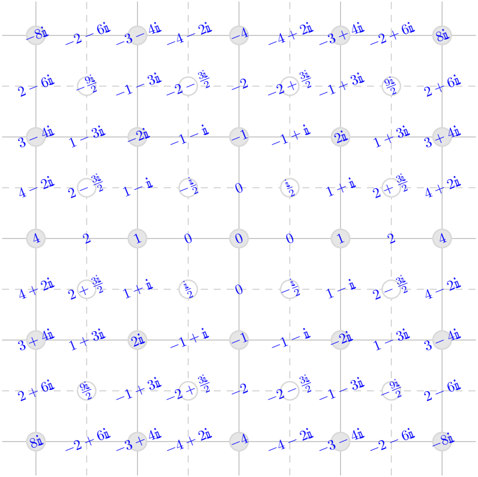
values of $z^{[0]}$

Discrete monomial functions (example 1)



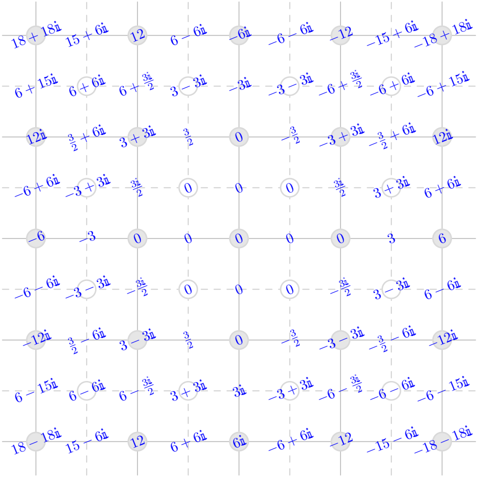
values of $z^{[1]}$

Discrete monomial functions (example 2)



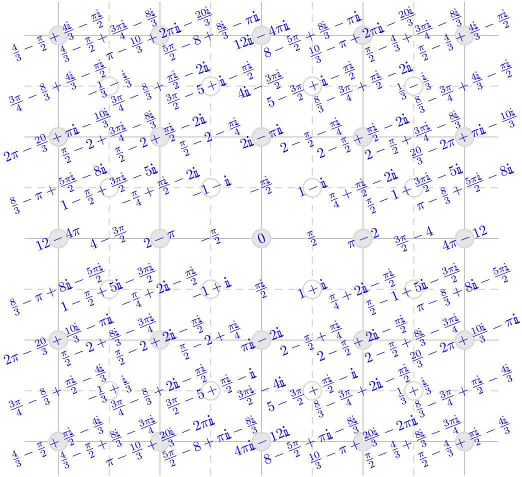
values of $z^{[2]}$

Discrete monomial functions (example 3)



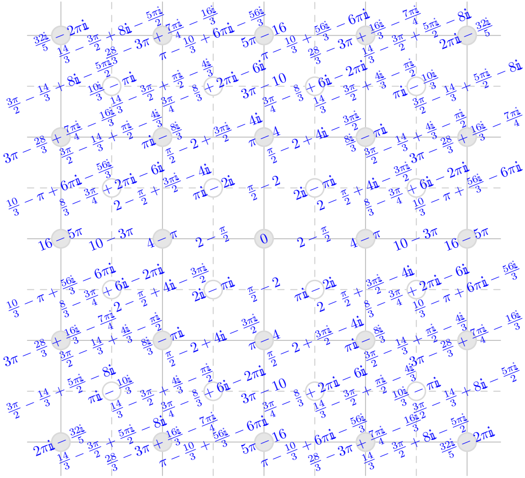
values of $z^{[3]}$

Discrete monomial functions (example -1)



values of $z^{[-1]}$

Discrete monomial functions (example -2)



values of $z^{[-2]}$

Outline of the proof of Theorems B and C

[Adame-Carrillo & Behzad & K., 2023^{+}]

Thm B: DGFF fields form a Fock space: $\mathcal{P}_\nabla/\mathcal{N} \cong \mathscr{F} \otimes \overline{\mathscr{F}}$

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0. $1 \in \mathcal{P}_\nabla$ satisfies $\mathsf{J}_\ell(1+\mathcal{N}) = 0$, $\bar{\mathsf{J}}_\ell(1+\mathcal{N}) = 0$ for $\ell \geq 0$, so

$$\mathcal{P}_\nabla/\mathcal{N} \hookleftarrow \mathscr{F} \otimes \overline{\mathscr{F}}.$$

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Thm C: $\mathrm{E}_{\Omega_\delta^\#}\Big[\prod_{j=1}^n \Big(\frac{F_j(z_j^\#)}{\delta^{\Delta_j+\bar{\Delta}_j}}\Big)\Big] \underset{\delta\to 0}{\longrightarrow} \langle F_1(z_1)\cdots F_n(z_n)\rangle_\Omega$

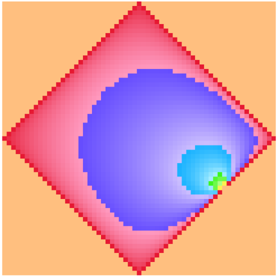
Wick's formula
$G^\#_{\Omega_\delta^\#}(\cdot,\cdot) \to G_\Omega(\cdot,\cdot)$

$\delta \oint\!\!\!\!\!\#(\cdots)\mathrm{d}^\# z \to \oint(\cdots)\mathrm{d}z$
$\frac{1}{\delta}\partial_\delta \to \partial, \quad \frac{1}{\delta}\bar{\partial}_\delta \to \bar{\partial}$
$\delta^n z_\#^n \to z^n, \quad \delta^n \bar{z}_\#^n \to \bar{z}^n$

Use the identification $\mathcal{F} \cong \mathscr{F} \otimes \overline{\mathscr{F}}$
to understand the L_0, $\bar{\mathrm{L}}_0$ eigenvalues in $\mathcal{F}$.

Linear local fields of DGFF

Linear local fields: $\mathcal{L} := \mathrm{span}\left\{ \varphi(z) - \varphi(0) \;\middle|\; z \in \mathbb{Z}^2 \right\} \subset \mathcal{P}_\nabla$



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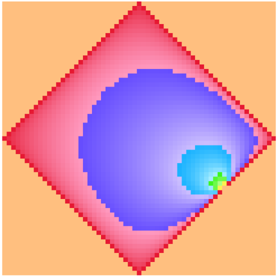
$\mathrm{R}(F) := \min_{\tilde{F} - F \in \mathcal{N}} \max \left\{\|x\|_1 \,\middle|\, x \in \mathrm{supp}\, \tilde{F}\right\}$ "radius of support of $F \in \mathcal{P}$"

Filtration by radius: $\mathcal{L}_{(1)} \subset \mathcal{L}_{(2)} \subset \cdots \subset \mathcal{L}_{(r)} \subset \cdots \subset \mathcal{L}$

$\mathcal{L}_{(r)} := \left\{L \in \mathcal{L} \,\middle|\, \mathrm{R}(L) \leq r\right\}$



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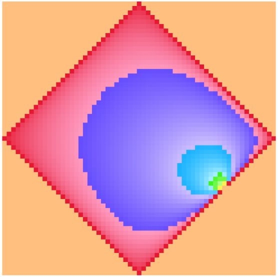
Obs A: $\dim \mathcal{L}_{(r)} / \mathcal{N} < \infty$



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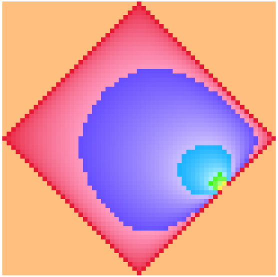

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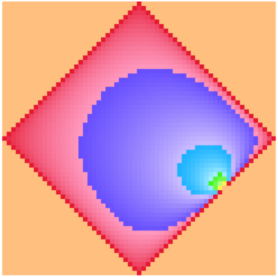
Obs B: $\dim \mathcal{L}_{(r)} / \mathcal{N} \ge 4r - 1$

$\mathrm{J}_{-k}\mathbf{1}, \bar{\mathrm{J}}_{-k}\mathbf{1} \in \mathcal{L}_{(r)}/\mathcal{N}$ for $1 \le k < 2r$

$\mathrm{J}_{-2r}\mathbf{1} + \bar{\mathrm{J}}_{-2r}\mathbf{1} \in \mathcal{L}_{(r)}/\mathcal{N}$



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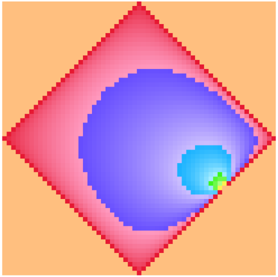

Cor: $\mathcal{L}/\mathcal{N} = \text{span}\left\{\text{J}_{-k}\mathbf{1}, \bar{\text{J}}_{-k}\mathbf{1} \;\middle|\; k > 0\right\}$

Higher degree local fields of DGFF

Normal ordered products: $\circ\!\!\circ \cdots \circ\!\!\circ \;:\; \bigoplus_{d\in\mathbb{N}} \mathcal{L}^{\otimes d} \to \mathcal{P}$

$$\circ\!\!\circ \prod_{i=1}^{d} \Phi_0(z_i) \circ\!\!\circ := \sum_{P} (-1)^{|P|} \prod_{j \notin \bigcup P} \Phi_0(z_j) \prod_{\{j,k\}\in P} C(z_j, z_k) \quad \text{(lin. ext.)}$$

$$C(z,w) := G_{\mathbb{Z}^2}(z,w) - G_{\mathbb{Z}^2}(z,0) - G_{\mathbb{Z}^2}(0,w) + G_{\mathbb{Z}^2}(0,0)$$

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Lemma: Normal ordered products factor through quotient(s) by (linear) null fields and are surjective:

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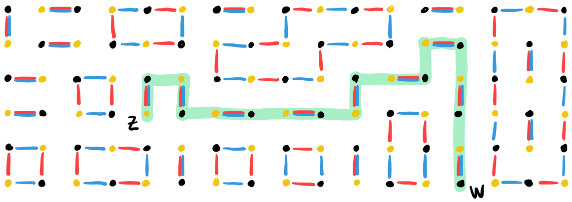
Thm: $\mathcal{F} \cong \mathscr{F} \otimes \overline{\mathscr{F}} \qquad (\mathcal{F} = \mathrm{span}\left\{ \mathrm{J}_{-k_m} \cdots \mathrm{J}_{-k_1} \bar{\mathrm{J}}_{-k'_{m'}} \cdots \bar{\mathrm{J}}_{-k'_1} \mathbf{1} \right\})$

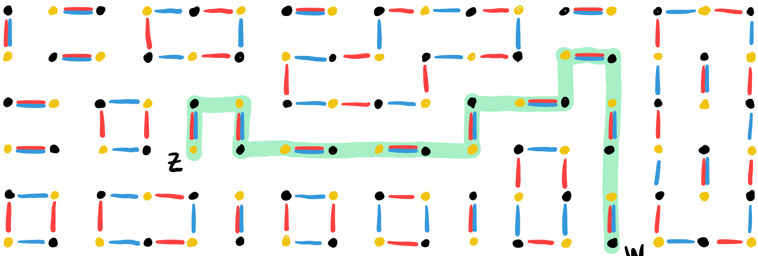
Perspectives

	discrete GFF	**Ising model**	**double dimer /** symplectic fermion	?
Virasoro ↷ local fields	$c = 1$ ✓ [HKV22]	$c = \frac{1}{2}$ ✓ [HKV22]	$c = -2$ ✓ [A23]	?
1-to-1 correspondece	✓ [ABK23+]	?	?	?
scaling limit correlations	✓ [ABK23+]	?	?	?



$$\eta(z)\xi(w)$$

Thank you!



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