The Ashkin–Teller model and the GFF

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Dimers

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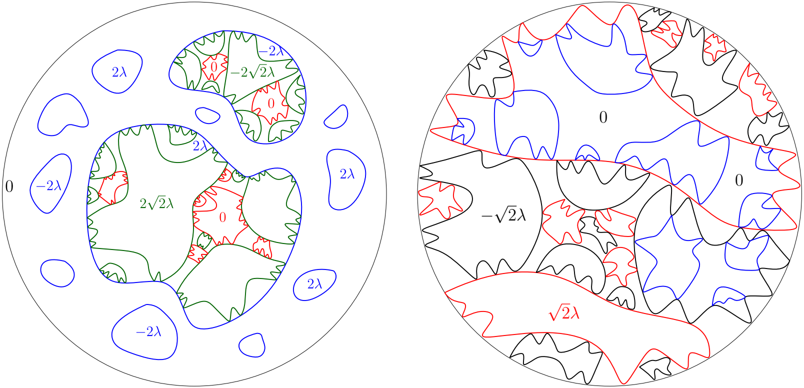
Outline



2λ
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Outline

1) The *double random current* model and main results
   - definition as an (enhanced) percolation model
   - the associated height function – *nesting field*
   - *two-valued sets* of the *Gaussian free field* (GFF)
   - Main theorem: *joint* convergence of the nesting field together with the *contours* of the critical DRC (both *outer and inner boundaries*) to a continuum GFF together with its certain two-valued sets
2) About the proof:
   - a discussion in the discrete and convergence of the nesting field to the GFF
   - tightness of interfaces, and some properties of subsequential limits
   - identification of the limit through properties of two-valued sets
3) A conjecture about the *Ashkin–Teller model* (*6V model*).

1) The double random current model and main results

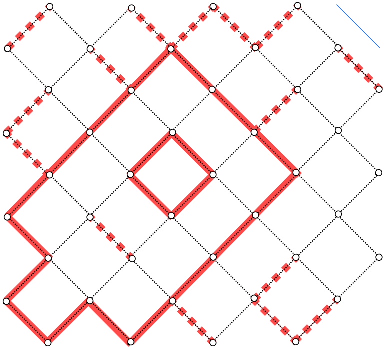
Currents

A *current* on a graph $G = (V, E)$ is a pair $\mathbf{n} = (\mathbf{n}_{\text{odd}}, \mathbf{n}_{\text{even}})$ satisfying

- $\mathbf{n}_{\text{odd}} \subseteq E$ is an *even subgraph* of G,
- $\mathbf{n}_{\text{even}} \subseteq E \setminus \mathbf{n}_{\text{odd}}$.

A connected component of

- $\mathbf{n}_{\text{odd}} \cup \mathbf{n}_{\text{even}}$ is called a *cluster*,
- $\mathbf{n}_{\text{odd}}$ is called an *interface*.



Double random currents

Let Ω be the set of currents.

For $\beta > 0$, define the *double random current* probability measure on Ω by

$$\mathbf{P}_{G,\beta}(\mathbf{n}) \propto 2^{k(\mathbf{n})} \sinh(2\beta)^{|\mathbf{n}_{\text{odd}}|} \big(\cosh(2\beta) - 1\big)^{|\mathbf{n}_{\text{even}}|},$$

where $k(\mathbf{n})$ is the *number of clusters* of $\mathbf{n}$.

We mostly focus on the *critical model* on the square lattice

$$\beta = \beta_c = \tfrac{1}{2} \ln(\sqrt{2} + 1),$$

and usually drop β them from the notation.

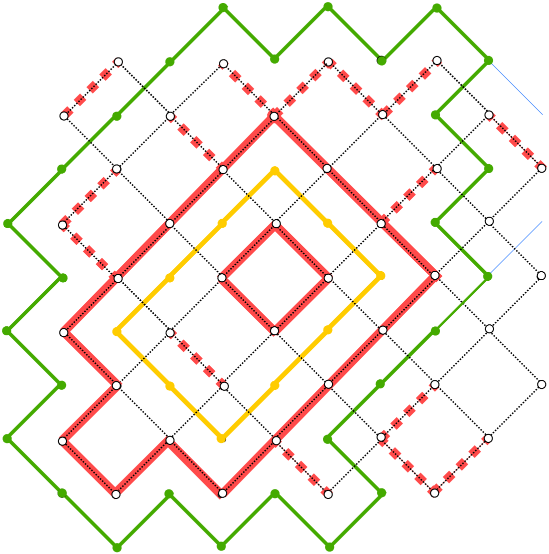
Double random currents

Double random currents are derived from the *Ising model* and posses a special combinatorial structure which is expressed in the celebrated *switching lemma* of Griffiths, Hurst and Sherman '70.

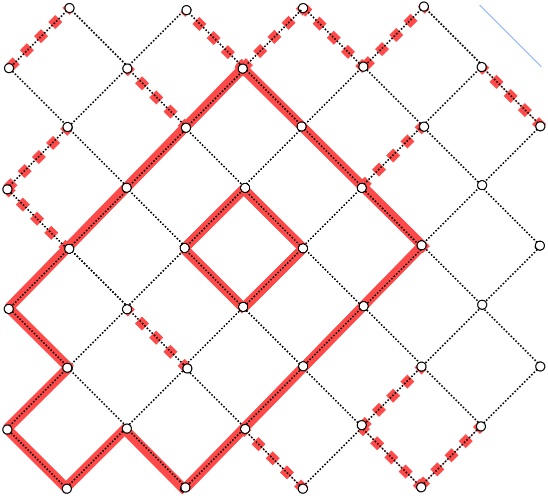
They were used by

- Aizenman '82 to prove triviality of the Ising field in dimension $d > 4$,
- Aizenman, Barsky and Fernandez '87 to obtain sharpness of phase transition for a general family of translation invariant spins systems,
- Aizenman, Duminil-Copin and Sidoravicius '14 to prove continuity of phase transition for Ising models on a large family of lattices including $\mathbb{Z}^3$,
- Aizenman and Duminil-Copin '20 to prove triviality in dimension $d = 4$

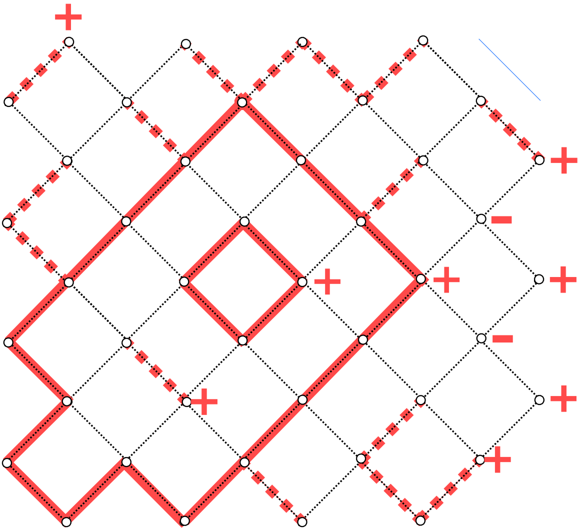
Inner and outer boundaries



Nesting field

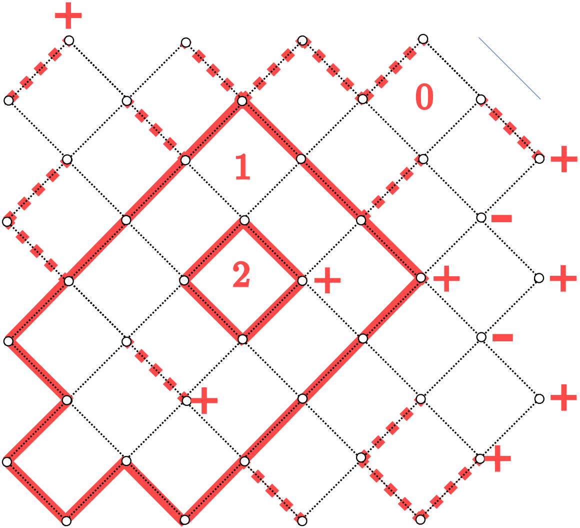


Nesting field



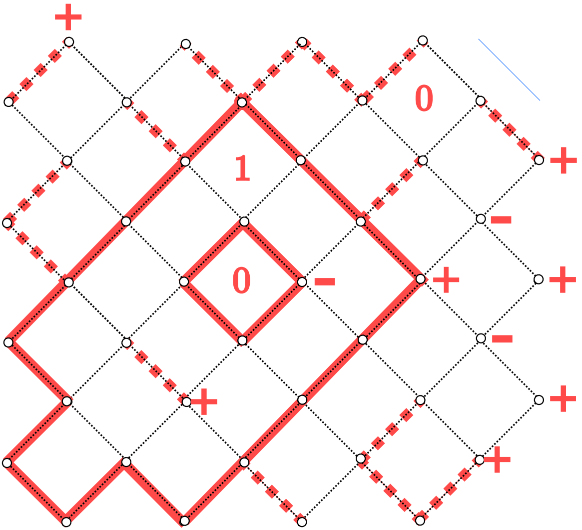
Independent spins for each *cluster*.

Nesting field



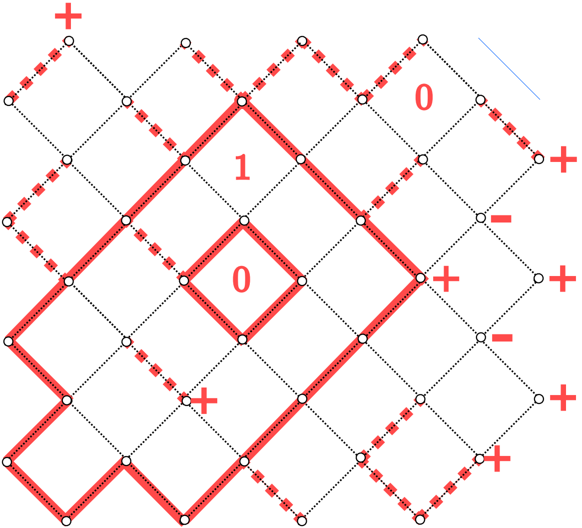
Heights change only across *contours*.
Sign of increment given by spin when crossing from outside to inside.

Nesting field



Heights change only across *contours*. Sign of increment given by spin.

Nesting field



For nested contours in the same cluster, increments *alternate* with each layer.

The Dirichlet Gaussian free field (GFF)

Let $D \subset \mathbb{C}$ be a domain with boundary and let

$$G_D(x,y) = \int_0^\infty p_t^D(x,y)dt$$

be the *Green's function* of Brownian motion killed upon hitting ∂D.

The *Gaussian free field with Dirichlet b.c.* h is a random *distribution* satisfying

- $h(f)$ is a mean-zero Gaussian for all test functions f,
- $$\mathbf{E}[h(f)h(g)] = \int_{\mathcal{D}} \int_{\mathcal{D}} f(x)g(y)G_D(x,y)dxdy$$

  for all f, g.

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  for all f, g.

The value at a point does not make sense!

Two-valued sets of the GFF

Introduced by Aru, Sepúlveda and Werner '17 and studied by Aru, Sepúlveda '18.

Heuristics

Let $a, b > 0$. Heuristically, the *two-valued level set* $\mathbb{A}_{-a,b}$ of the GFF h, is the set of points in D that are connected to the boundary ∂D by a path of points where h takes values in $[-a, b]$.

$\mathbb{A}_{-a,b}$ exists iff $a + b \geq 2\lambda$, where $\lambda = \sqrt{\pi/8}$.

$\mathbb{A}_{-a,b}$ is measurable wrt to h.

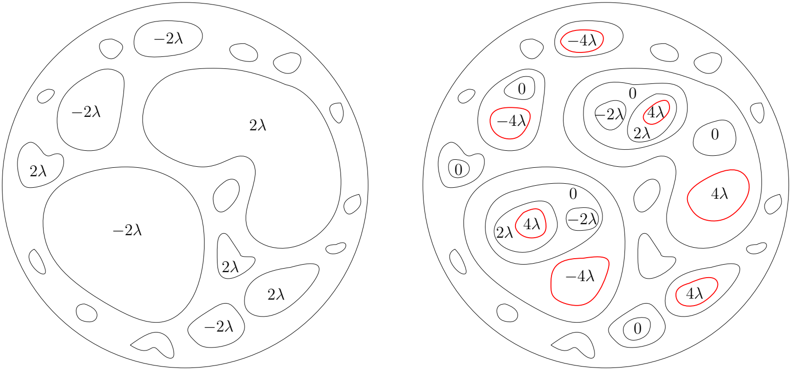
Let $\mathcal{L}_{-a,b}$ be the set of boundaries of the conn. components of $D \setminus \mathbb{A}_{-a,b}$.

A loop $\ell \in \mathcal{L}_{-a,b}$ with *label* $-a$ touches ∂D, iff $a < 2\lambda$ and $\ell \in \mathcal{L}_{-a,2\lambda-a}$.

Coupling of CLE_4 and GFF

Theorem (Miller & Sheffield ’11)

CLE_4 has the same distribution as $\mathcal{L}_{-2\lambda,2\lambda}$. Moreover, the *nesting field* of the CLE_4 has the same distribution as the GFF.



Convergence of DRC with free boundary conditions

Fix a simply connected bounded Jordan domain $D \subset \mathbb{C}$, and let $D^\delta = D \cap \delta\mathbb{Z}^2$ be its approximation.

Let $\mathbf{n}^\delta$ and h^δ be the associated critical double random current and its nesting field.

Let B^δ be the collection of outer boundaries of $\mathbf{n}^\delta$.

For each loop $\ell^\delta \in B^\delta$, let $A^\delta(\ell^\delta)$ be the collection of loops corresponding to the inner boundary of the cluster whose outer boundary is ℓ^δ.

Let $A^\delta := \cup_{\ell^\delta \in B^\delta} A^\delta(\ell^\delta)$.

For a loop ℓ, let $O(\ell)$ be the domain encircled by ℓ.

Convergence of DRC with free boundary conditions

Theorem (Duminil-Copin & L. & Qian '21)

As $\delta \to 0$, the law of $(h^\delta, B^\delta, A^\delta)$ under $\mathbf{P}_{D^\delta}$ converges to $(\frac{1}{2\sqrt{2}\lambda}h, B, A)$, where

- h is a GFF in D.
- $B = \mathrm{CLE}_4(h) = \mathcal{L}_{-2\lambda,2\lambda}(h)$.
- If the outer boundary ℓ^δ of a cluster converges to a loop $\ell \in B$ with boundary value 2λ (resp. -2λ), then $A^\delta(\ell^\delta)$ converges to $\mathcal{L}_{-2\lambda,(2\sqrt{2}-2)\lambda}(h|_{O(\ell)})$ (resp. $\mathcal{L}_{(2-2\sqrt{2})\lambda,2\lambda}(h|_{O(\ell)})$).

Moreover

- If a loop ℓ^δ in A^δ converges to ℓ, then the outer boundaries of the outermost clusters enclosed by ℓ^δ converge to a $\mathrm{CLE}_4(h|_{O(\ell)})$.

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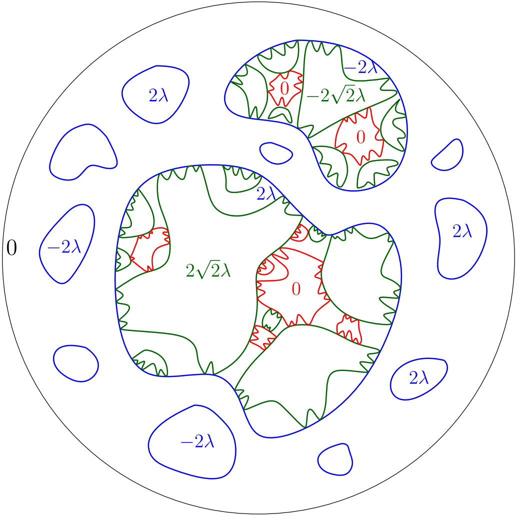
The gap 2λ does not exist on the discrete level!

Convergence of DRC with free boundary conditions



Duality

Let $G^\dagger$ to be the dual graph and dual (including the vertex corresponding to the unbounded face of G).

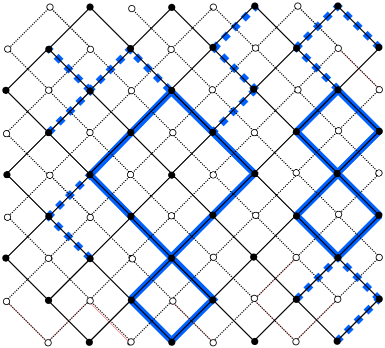
The *dual* double random current model is the probability measure $\mathbf{P}_{G^\dagger,\beta^\dagger}$, with

$$\exp(-2\beta^\dagger) = \tanh(\beta).$$

The critical temperature is *self-dual*:

$$\beta_c^\dagger = \beta_c.$$

The nesting field is analogous but takes values in $\mathbb{Z} + \frac{1}{2}$.



Convergence of DRC with wired boundary conditions

Let D and D^δ be as before.

Let $(D^\delta)^\dagger$ be the dual graph (the outer *ghost* vertex has large degree).

Let $\mathbf{n}^\delta$ and h^δ be the critical double random current and its nesting field on $(D^\delta)^\dagger$.

Let $\widehat{A}^\delta$ be the collection of loops in the inner boundary of the cluster of the ghost vertex.

Convergence of DRC with wired boundary conditions

Theorem (Duminil-Copin & L. & Qian '21)

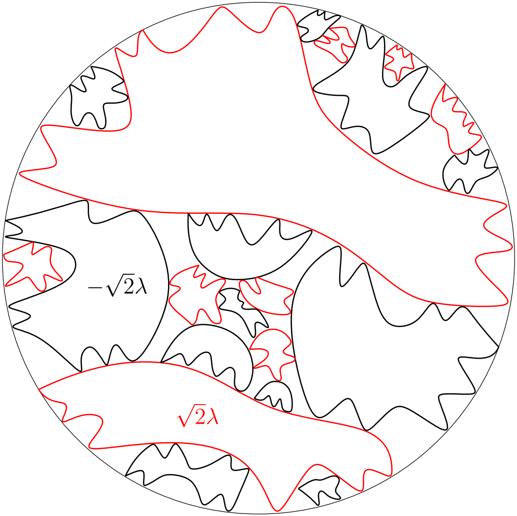
As $\delta \to 0$, the law of $(h^\delta, \widehat{A}^\delta)$ under $\mathbf{P}_{(D^\delta)^\dagger}$ converges to $(\frac{1}{2\sqrt{2}\lambda} h, \widehat{A})$, where

- h is a GFF in D.
- $\widehat{A} = \mathcal{L}_{-\sqrt{2}\lambda, \sqrt{2}\lambda}(h)$.

Moreover,

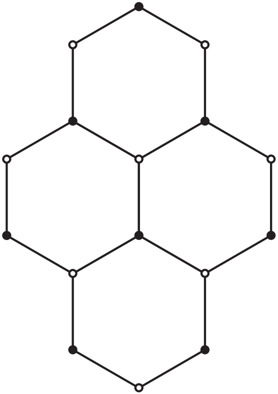
- In each inner boundary loop $\ell^\delta \in \widehat{A}^\delta$, the $\mathbf{n}^\delta$ has free boundary conditions, and the previous theorem applies.

Convergence of DRC with wired boundary conditions

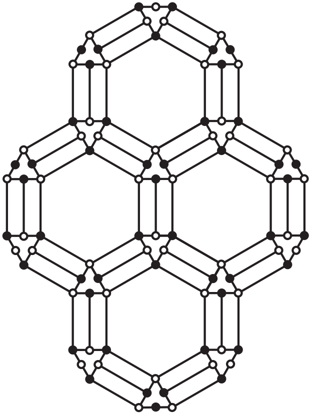


2) About the proof

Connection to dimers



G



G^d

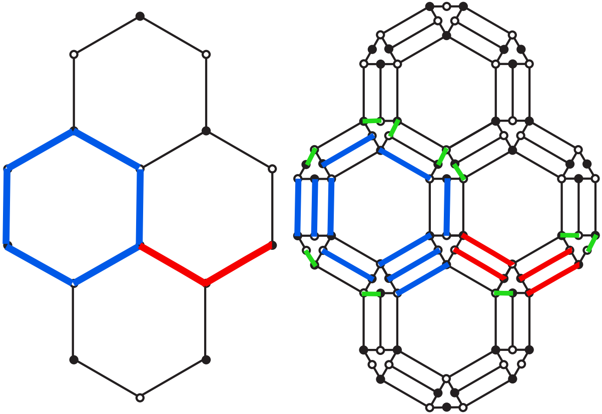
Connection to dimers

Note that the set of faces of G embeds naturally in the set of faces of G^d.

Theorem (Duminil-Copin & L., 2016)

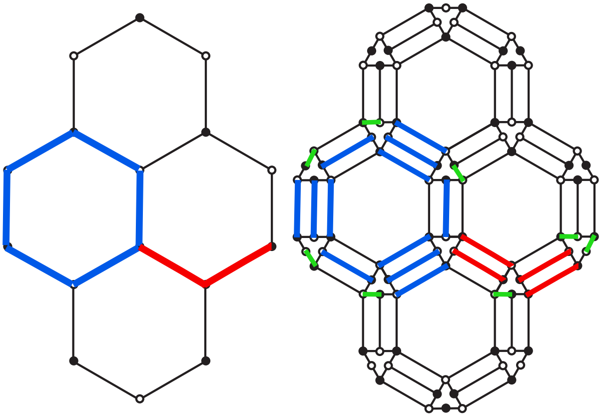
Under this mapping, the height function on G^d restricted to the faces of G becomes the double random current nesting field.

A measure preserving mapping - edge factors



$$\mathbf{P}_{G,\beta}(\mathbf{n}) \propto 2^{k(\mathbf{n})} \sinh(2\beta)^{|\mathbf{n}_{\text{odd}}|} \big(\cosh(2\beta) - 1\big)^{|\mathbf{n}_{\text{even}}|}$$

A measure preserving mapping - edge factors



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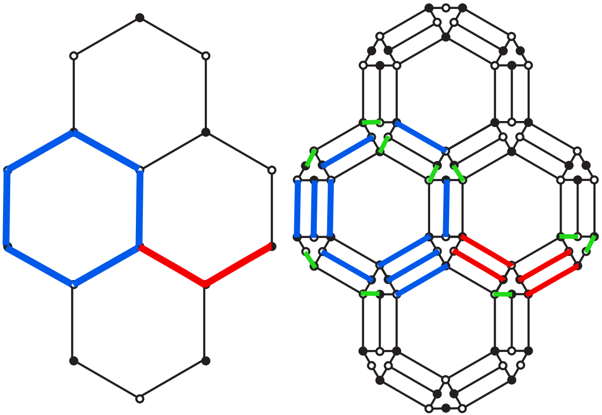

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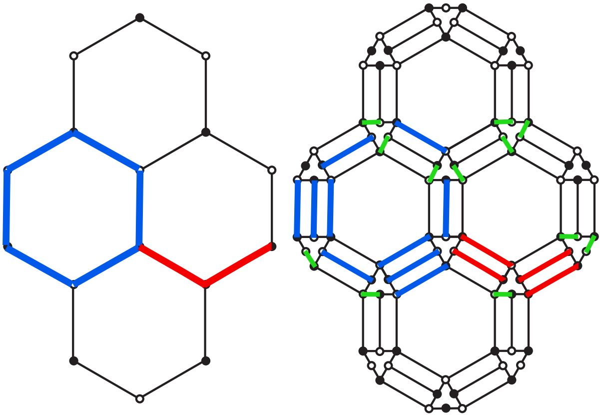
A measure preserving mapping - vertex factors



$$\mathbf{P}_{G,\beta}(\mathbf{n}) \propto 2^{k(\mathbf{n})} \sinh(2\beta)^{|\mathbf{n}_{\mathrm{odd}}|} \big(\cosh(2\beta) - 1\big)^{|\mathbf{n}_{\mathrm{even}}|}$$

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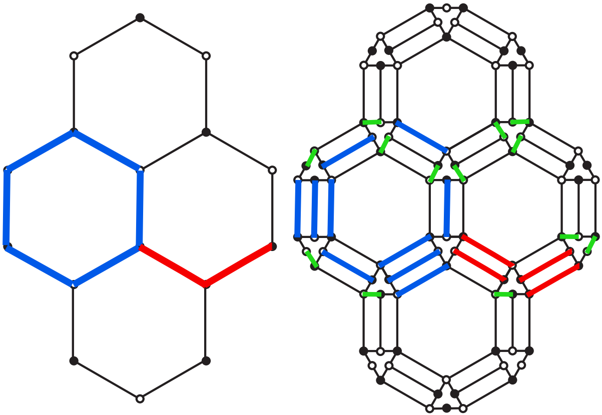

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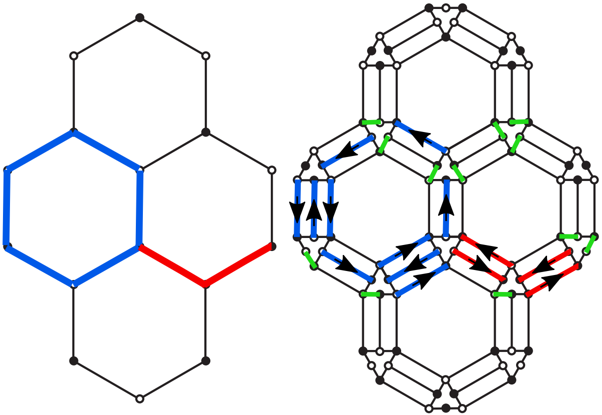
A measure preserving mapping - cluster factors



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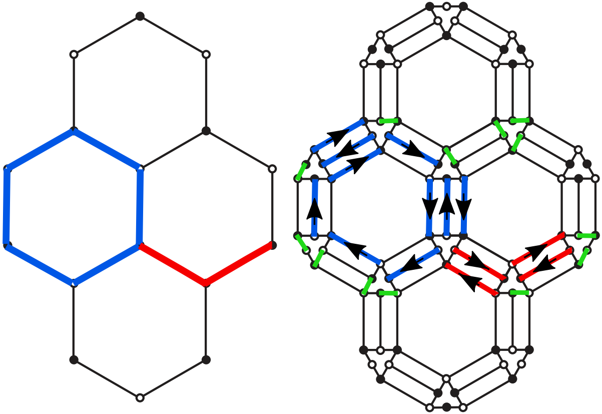

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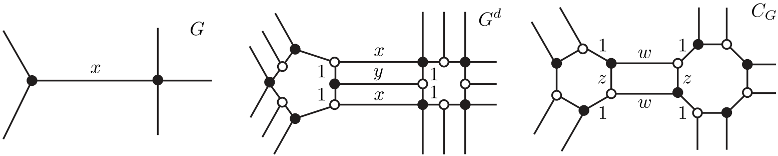
Connection to dimers

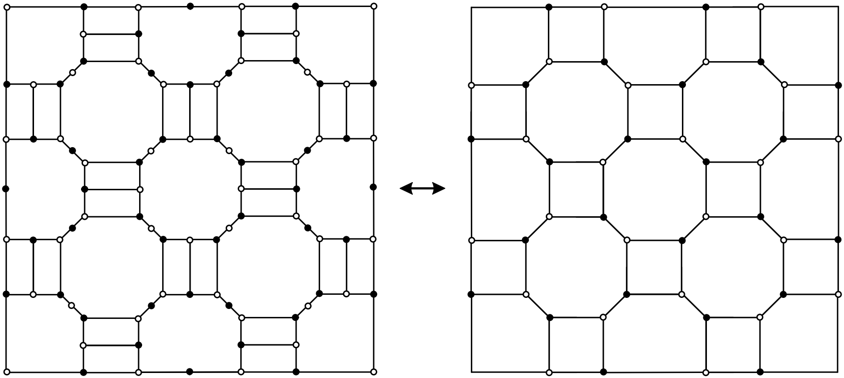
Dubédat (2011) (based on Fan and Wu (1970)) provided a mapping between the double Ising model on a graph G and dimers on a related graph C_G.

Boutillier and de Tilière (2012) provided a different proof of the same mapping and showed convergence to full plane GFF.



The weights satisfy $y = \frac{2x}{1-x^2}$, $w = \frac{2x}{1+x^2}$, $z = \frac{1-x^2}{1+x^2}$, where $x = \tanh \beta$.

Connection to dimers



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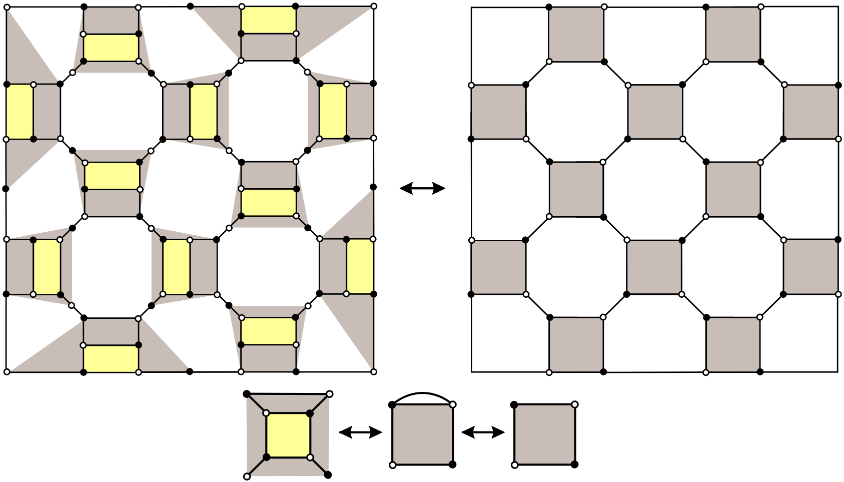


Scaling limit of the height function

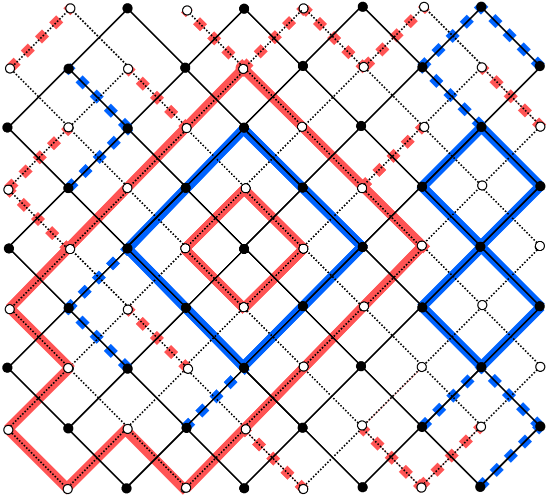
Theorem (Duminil-Copin & L. & Qian, ’21)

As $\delta \to 0$, the critical double random current nesting field drawn according to $\mathbf{P}_{D^\delta}$ or $\mathbf{P}_{(D^\delta)^+}$ converges to $\frac{1}{2\sqrt{2}\lambda} h$ where h is a GFF in D.

Proof

- Express the inverse Kasteleyn matrix K^{-1} on C_G in terms of the spin fermionic observable of Chelkak and Smirnov ’09, and Hongler and Smirnov ’10.
- Use the scaling limit of fermionic observables to identify the moments of the height function at macroscopic distances (like in Kenyon ’99).
- Use crossing estimates to control the moments at short distances.

Master coupling



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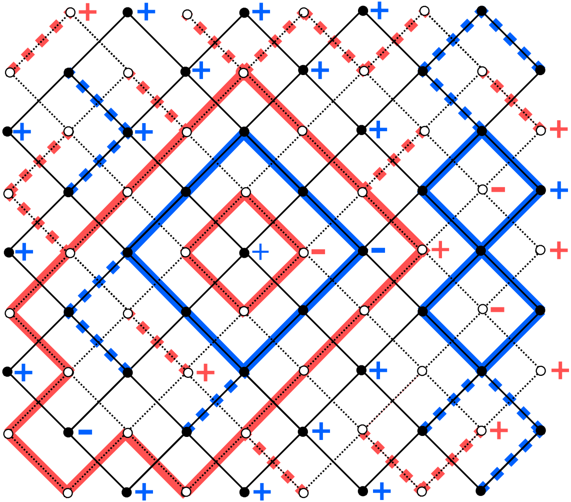

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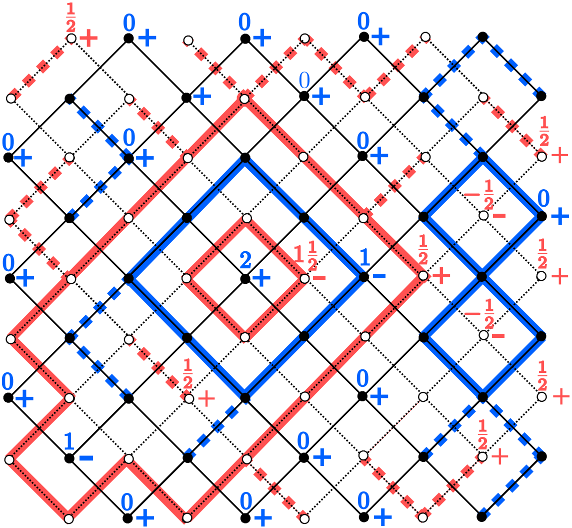
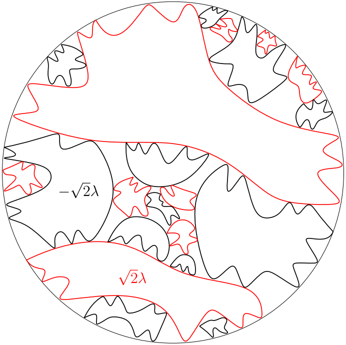


Identification of the limit – wired boundary conditions

Theorem (Duminil-Copin & L. & Qian, '21)

As $\delta \to 0$, the law of $(h^\delta, \widehat{A}^\delta)$ under $\mathbf{P}_{(D^\delta)^\dagger}$ converges to

$$\left(\frac{1}{2\sqrt{2}\lambda} h, \mathcal{L}_{-\sqrt{2}\lambda, \sqrt{2}\lambda}(h)\right).$$



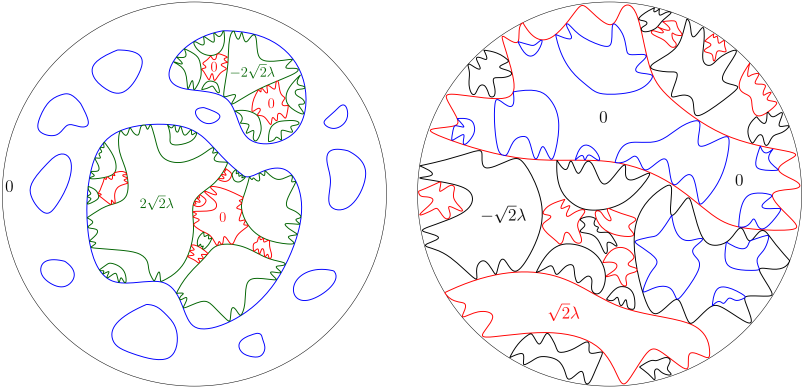
Identification of the limit – free boundary conditions

Proposition (Duminil-Copin & L. & Qian ’21)

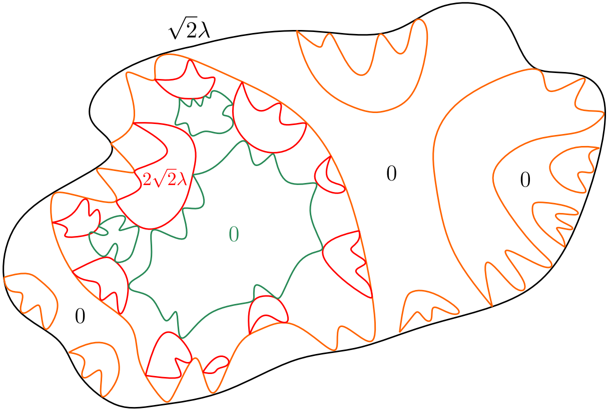
The family $(h^\delta, B^\delta, A^\delta)_{\delta>0}$ is tight and for every subsequential limit (h, B, A), a.s.

1. h is a GFF in D.
2. The sets A and B consist of simple loops which do not cross each other. Every loop in A is encircled by some loop in B.
3. Almost surely, any two loops in B do not intersect each other.
4. *(Local set)* The gasket of A is a *thin local set* of h with boundary values belonging to
$$\{-2\sqrt{2}\lambda, 0, 2\sqrt{2}\lambda\}.$$
More precisely, for each loop $\ell \in B$, for all $\gamma \in A(\ell)$, h restricted to $O(\gamma)$ is equal to an independent GFF with boundary condition 0 or $\pm 2\sqrt{2}\lambda$.
5. The loops in A that have boundary value 0 do not touch the loops in B.

Identification of the limit – free boundary conditions



Identification of the limit – free boundary conditions



3) A conjecture on the Ashkin–Teller model

A conjecture on the Ashkin–Teller model

For $\beta > 0$ and $U \in \mathbb{R}$, *Ahskin–Teller currents* are given by

$$\mathbf{P}_{G,\beta}(\mathbf{n}) \propto 2^{k(\mathbf{n})}\big(e^{2U}\sinh(2\beta)\big)^{|\mathbf{n}_{\text{odd}}|}\big(e^{2U}\cosh(2\beta)-1\big)^{|\mathbf{n}_{\text{even}}|},$$

Conjecture (L. '21)

Consider the *critical Ashkin–Teller model* of the square lattice model given by

$$\sinh(2\beta) = e^{-2U}, \qquad \beta \geq U.$$

Then, analogous theorems as for $U = 0$, hold but with $\sqrt{2}$ replaced by $\sqrt{g}$ in all the statements, where g satisfies

$$\sin\left(\frac{\pi}{8}g\right) = \coth(2\beta)/2.$$

Thank you for your attention!