Universality for multi-component stochastic systems

Alessandra Occelli

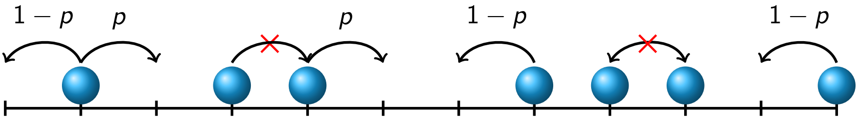
joint work with G. Cannizzaro, P. Gonçalves, R. Misturini

arXiv:2304.02344

Université d'Angers

Dimers 2023
Paris 10.07.2023

The simple exclusion process



- at most one particle at each $x \in \mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$: $\eta_t(x) \in \{0,1\}$, $x \in \mathbb{T}_N$
- particles exchange position on $\mathbb{T}_N$ with rate 1 and transition probability p:
  $p = 1/2$ SSEP, $p \neq 1/2$ ASEP, $p = 1$ TASEP, $p \sim 1/N^{\gamma}$, $\gamma \geq 0$ WASEP
- number of particles $\sum_{x\in\mathbb{T}_N} \eta_t(x)$ conserved
- invariant measures = Bernoulli product of parameter $\varrho \in (0,1)$

Hydrodynamic limit

$$\pi_t^N(du) = \frac{1}{N} \sum_{x\in\mathbb{T}_N} \eta_{tN^a}(x)\delta_{\frac{x}{N}}(du)$$

can evolve as the heat equation, the inviscid Burgers' equation, the viscous Burgers' equation,. . .

Equilibrium fluctuations

$$\mathcal{Y}_t^N(f) = \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} f\left(\tfrac{x}{N}\right)(\eta_{tN^a}(x) - \varrho)$$

can converge to the solution of the Ornstein–Uhlenbeck, of the stochastic Burgers' equation,. . .

Universality for one-component systems

Let $h(t,u)$ be a stochastic process (called *height function*)

Consider the space-time renormalization group operator with exponents $1 : 1/r : z/r$

$$\mathfrak{R}_\varepsilon h(t,u) = \varepsilon^{-1} h(t\varepsilon^{z/r}, u\varepsilon^{-1/r})$$

UNIVERSALITY CLASS: basin of attraction of $\lim_{\varepsilon\to 0} \mathfrak{R}_\varepsilon h$

For SEP:

- Edward–Wilkinson (diffusive) 1:2:4
- Kardar–Parisi–Zhang (super-diffusive) 1:2:3

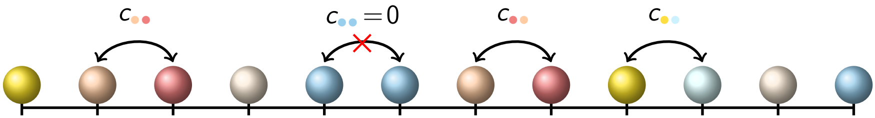
KPZE/SBE

EW ⟷ KPZ

$$\partial_t h_t = \frac{1}{2}\partial^2_{xx} h_t + \frac{1}{2}(\partial_x h_t)^2 + \xi_t$$

$$\partial_t \mathcal{Y}_t = \frac{1}{2}\partial^2_{xx}\mathcal{Y}_t + \frac{1}{2}\partial_x \mathcal{Y}_t^2 + \partial_x \xi_t$$

Universality for multi-component systems



Consider a system with M conserved quantities $\vec{\varrho} = (\varrho_1, \ldots, \varrho_M)$

Different **universality classes** can appear, identified by *exponents* and *scaling functions* characterizing the macroscopic fluctuations

To see which universality classes might appear, one can use the approach of **Nonlinear Fluctuating Hydrodynamic Theory** (NLFHT) Spohn 2014 describing the fluctuations of the conserved quantities of a multi-component system in terms of SPDE

Nonlinear Fluctuating Hydrodynamic Theory (NLFHT)

1. Hydrodynamic equations: $\partial_t \vec{\varrho}(t,u) + \nabla \vec{j}(t,u) = 0$
2. Express the current $j_\alpha(\vec{\varrho})$, $\alpha = 1, \ldots, M$, as a function of $\vec{\varrho}$:

$$\partial_t \vec{\varrho}(t,u) + J(t,u)\nabla \vec{\varrho}(t,u) = 0$$

3. Add a phenomenological diffusion matrix $D\Delta\vec{\varrho}(t,u)$ and white noise terms $B\vec{\xi}$
4. Expand $\rho_\alpha(t,u) = \rho_\alpha + Y_\alpha(t,u)$ to second order: the fluctuation fields satisfy

$$\partial_t \vec{Y} = -\nabla\left\{ J\vec{Y} + \frac{1}{2}\sum_{\alpha=1}^{M} \vec{Y}^T H^\alpha \vec{Y} + D\nabla\vec{Y} + B\vec{\xi} \right\}$$

5. Transform the fields $\vec{Y}$ into normal fields $\vec{\phi} = R\vec{Y}$, where $RJR^{-1} = diag(v_\alpha)$:

$$\partial_t \phi_\alpha = -\nabla\left\{ v_\alpha \phi_\alpha + \vec{\phi}^T G^\alpha \vec{\phi} + (\tilde{D}\nabla\vec{\phi})_\alpha + (\tilde{B}\vec{\xi})_\alpha \right\}$$

where $G^\alpha = \frac{1}{2}\sum_{\beta=1}^{M} R_{\alpha\beta}(R^{-1})^T H^\beta R^{-1}$ are the **mode coupling matrices**

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5. Transform the fields $\vec{Y}$ into normal fields $\vec{\phi} = R\vec{Y}$, where $RJR^{-1} = diag(v_\alpha)$:

$$\partial_t \phi_\alpha = -\nabla \Big\{ \overset{\text{drift}}{v_\alpha \phi_\alpha} + \vec{\phi}^T G^\alpha \vec{\phi} + (\tilde{D}\nabla\vec{\phi})_\alpha + (\tilde{B}\vec{\xi})_\alpha \Big\}$$

where $G^\alpha = \frac{1}{2}\sum_{\beta=1}^{M} R_{\alpha\beta}(R^{-1})^T H^\beta R^{-1}$ are the **mode coupling matrices**

Nonlinear Fluctuating Hydrodynamic Theory (NLFHT)

6. The universal large-scale behavior is encoded in the **dynamical structure function**

$$S^{\alpha\beta}(t,u) = \langle \phi_\alpha(t,u)\phi_\beta(0,0)\rangle$$

7. When $v_\alpha \neq v_\beta$, $\alpha \neq \beta$, off-diagonal terms decay quickly in t and u
8. The behavior of diagonal terms is universal:

$$S^{\alpha\alpha}(t,u) \sim t^{-1/z_\alpha} f_\alpha((u - v_\alpha)^{z_\alpha}/t)$$

9. The scaling functions f_α and the dynamical exponents z_α can be evaluated using mode coupling matrices G^α
10. There exists an infinite family of universality classes with $z_\alpha = F_{\alpha+2}/F_{\alpha+1}$, where F_α are Fibonacci numbers:

$$z_\alpha = 2,\ 3/2,\ 5/3,\ 8/5,\ \dots, \varphi$$

Nonlinear Fluctuating Hydrodynamic Theory (NLFHT)

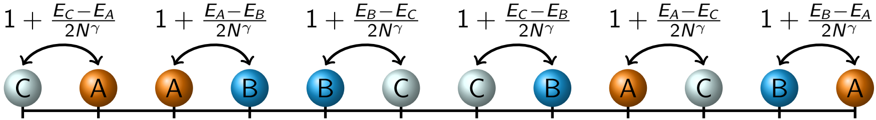
Details for 2 conserved quantities (Popkov–Schmidt–Schütz 2015):

$G^2 \backslash G^1$	$\begin{pmatrix} \star & \\ & \bullet \end{pmatrix}$	$\begin{pmatrix} \star & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \\ & \bullet \end{pmatrix}$	$\begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$
$\begin{pmatrix} \bullet & \\ & \star \end{pmatrix}$	(KPZ, KPZ)	(KPZ, KPZ)	($\frac{5}{3}$L, KPZ)	(D, KPZ')
$\begin{pmatrix} 0 & \\ & \star \end{pmatrix}$	(KPZ, KPZ)	(KPZ, KPZ)	($\frac{5}{3}$L, KPZ)	(D, KPZ)
$\begin{pmatrix} \bullet & \\ & 0 \end{pmatrix}$	(KPZ, $\frac{5}{3}$L)	(KPZ, $\frac{5}{3}$L)	(GM, GM)	(D, $\frac{3}{2}$L)
$\begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$	(KPZ', D)	(KPZ, D)	($\frac{3}{2}$L, D)	(D, D)

D = Gaussian universality class
KPZ = superdiffusive KPZ universality class
KPZ' = modified KPZ
z_αL = superdiffusive universality class with z_α-stable Lévy scaling function
GM = φL with the golden mean $\varphi = \frac{1+\sqrt{5}}{2}$

The ABC model

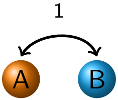
Exclusion process on $\mathbb{T}_N$ with three species of particles and (weakly asymmetric) rates



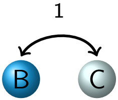
- Total numbers of particles A and B are conserved
- Invariant measures: product measure ν_ϱ given by $\nu_\varrho(\eta(x) = \alpha) = \varrho_\alpha$ for $x \in \mathbb{T}_N$ and $\alpha \in \{A, B, C\}$

Two subcases of interest:

$E_A = E_B$



$E_B = E_C$



Predictions

In the ABC model $G^1_{22} = G^2_{11} = 0 \Rightarrow$ KPZ or diffusive modes

$G^2 \backslash G^1$	$\begin{pmatrix} g_1 & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \\ & g_2 \end{pmatrix}$	(KPZ, KPZ)	(D, KPZ)
$\begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$	(KPZ, D)	(D, D)

$$g_1 = -\frac{1}{6N^\gamma}[-c_+^2 + (E_B - E_C)(c_+ - c_-) + c_+ c_-]$$
$$g_2 = -\frac{1}{6N^\gamma}[-c_-^2 + (E_B - E_C)(c_- - c_+) + c_+ c_-]$$

i $E_A = E_B \Rightarrow$ 1 mode KPZ, 1 mode diffusive

ii $E_B = E_C \Rightarrow$ 1 mode KPZ, 1 mode diffusive

iii $E_A \neq E_B \neq E_C \Rightarrow$ 2 modes KPZ

The fluctuation fields

The occupation number of the species $\alpha \in \{A, B, C\}$ is $\xi_x^\alpha(\eta) = \begin{cases} 1 & \eta(x) = \alpha \\ 0 & otherwise. \end{cases}$

The density fluctuation fields $\mathcal{Y}_t^N = (\mathcal{Y}_t^{N,A}, \mathcal{Y}_t^{N,B}, \mathcal{Y}_t^{N,C})$ is defined as

$$\mathcal{Y}_t^{N,\alpha}(du) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{T}_N} [\xi_x^\alpha(\eta_t) - \varrho_\alpha]\, \delta_{\frac{x}{N}}(du).$$

We consider the following fields

i $E_A = E_B \;\Rightarrow$
$$\mathcal{Z}_t^{N,+}(f) = \mathcal{Y}_t^{N,A}(T_{v_+ N^2 t} f) - \mathcal{Y}_t^{N,B}(T_{v_+ N^2 t} f)$$
$$\mathcal{Z}_t^{N,-}(f) = \mathcal{Y}_t^{N,A}(T_{v_- N^2 t} f) + \mathcal{Y}_t^{N,B}(T_{v_- N^2 t} f)$$
with $v_\pm = \pm \frac{(E_A - E_C)}{3N^\gamma}$

ii $E_B = E_C \;\Rightarrow$
$$\mathcal{Z}_t^{N,+}(f) = \mathcal{Y}_t^{N,A}(T_{v_+ N^2 t} f) + 2\mathcal{Y}_t^{N,B}(T_{v_+ N^2 t} f)$$
$$\mathcal{Z}_t^{N,-}(f) = \mathcal{Y}_t^{N,A}(T_{v_- N^2 t} f)$$
with $v_\pm = \pm \frac{(E_C - E_A)}{3N^\gamma}$

iii $E_A \neq E_B \neq E_C \;\Rightarrow$
$$\mathcal{Z}_t^{N,+}(f) = \mathcal{Y}_t^{N,A}(T_{v_+ N^2 t} f) + c_- \mathcal{Y}_t^{N,B}(T_{v_+ N^2 t} f)$$
$$\mathcal{Z}_t^{N,-}(f) = \mathcal{Y}_t^{N,A}(T_{v_- N^2 t} f) + c_+ \mathcal{Y}_t^{N,B}(T_{v_- N^2 t} f)$$

with $v_\pm = \pm \frac{\delta}{2N^\gamma}$, $c_\pm = (E_A - E_B \pm \frac{3}{2}\delta)/(E_A - E_C)$ and
$\delta = \frac{2}{3}\sqrt{(E_A - E_C)^2 + (E_B - E_C)^2 - (E_A - E_C)(E_B - E_C)}$

Energy solution to the stochastic Burgers' equation

$$d\mathscr{Z}_t^{\pm} = \Delta \mathscr{Z}_t^{\pm} dt + \lambda_{\pm} \nabla (\mathscr{Z}^{\pm})_t^2 + \sqrt{2\sigma_{\pm}^2} \nabla d\mathscr{W}_t^{\pm} \qquad \text{(SBE)}$$

To make sense to the non-linearity, we define it as the limit as $\varepsilon \to 0$ of the process $\{\mathscr{B}_t^{\varepsilon}; t \in [0, T]\}$:

$$\mathscr{B}_t^{\varepsilon}(\varphi) := \int_0^t \int_{\mathbb{R}} (\mathscr{Z}_s * \imath_{\varepsilon}(u))^2 \nabla \varphi(u) \, du \, ds$$

and ask that $\mathscr{Z}_t$ satisfies an *energy estimate*, i.e. $\exists C > 0$ s. t. $\forall s \leq t \in [0, T]$, $0 < \delta \leq \varepsilon < 1$ and $f \in \mathscr{D}(\mathbb{T})$, we have

$$\mathbb{E}\left[\left((\mathscr{B}_s^{\varepsilon}(\varphi) - \mathscr{B}_t^{\varepsilon}(\varphi)) - (\mathscr{B}_s^{\delta}(\varphi) - \mathscr{B}_t^{\delta}(\varphi))\right)^2\right] \leq C\varepsilon(t-s)\|\nabla\varphi\|_2^2$$

Definition $\{\mathscr{Z}_t; t \in [0, T]\}$ is a *stationary energy solution* of (SBE) if

- $\forall t \in [0, T]$ the $\mathscr{D}'(\mathbb{T})$-valued rv $\mathscr{Z}_t$ is a white noise of covariance σ^2;
- the process $\{\mathscr{Z}_t; t \in [0, T]\}$ satisfies an energy estimate;
- for any $\varphi \in \mathscr{D}(\mathbb{T})$ and $t \in [0, T]$, the process

  $$\mathscr{M}_t(\varphi) = \mathscr{Z}_t(\varphi) - \mathscr{Z}_0(\varphi) - \int_0^t \mathscr{Z}_s(\Delta\varphi) ds + \lambda \mathscr{B}_t(\varphi)$$

  is a continuous martingale wrt the natural filtration of $\mathscr{Z}_\cdot$, of qv $\langle \mathscr{M}(\varphi) \rangle_t = t2\sigma^2 \|\nabla\varphi\|_{L^2(\mathbb{T})}$
- the reversed processes $\{(\mathscr{Z}_{T-t}, \mathscr{B}_{T-t} - \mathscr{B}_T); t \in [0, T]\}$ also satisfy (iii) with λ replaced by $-\lambda$.

Main result

Theorem (Cannizzaro–Gonçalves–Misturini–O. '23)
In the diffusive time scaling, the sequence of processes $(\mathscr{X}^{N,+}, \mathscr{X}^{N,-})_{N\in\mathbb{N}}$ converges in law in $\mathcal{C}([0,T], \mathscr{D}'(\mathbb{T})^2)$ to $(\mathscr{X}^+, \mathscr{X}^-)$, where $\mathscr{X}^+$ and $\mathscr{X}^-$ are uncorrelated stationary solutions of stochastic Burgers' equations of the form

$$d\mathscr{X}_t^{\pm} = \Delta\mathscr{X}_t^{\pm}dt + \lambda_{\pm}\nabla(\mathscr{X}^{\pm})_t^2 + \sqrt{2\sigma_{\pm}^2}\nabla d\mathscr{W}_t^{\pm},$$

where $\mathscr{W}^+$ and $\mathscr{W}^-$ are independent $\mathscr{D}(\mathbb{T})'$-valued Brownian motions with covariances given on $f, g \in \mathscr{D}(\mathbb{T})$ by $\mathbb{E}[\mathscr{W}_t(\psi)\mathscr{W}_t(\varphi)] = (t\wedge s)\langle\psi,\varphi\rangle_{L^2(\mathbb{T})}$ and the coefficients $\lambda_{\pm}$ and $\sigma_{\pm}^2$ are given by

Case i-ii: $\sigma_+^2 = \frac{2}{3}$ *and* $\sigma_-^2 = \frac{2}{9}$.

- *when* $\gamma > 1/2$, $\lambda_+ = \lambda_- = 0$, *so that* $\mathscr{X}^{\pm}$, *are the unique solutions to the corresponding Ornstein–Uhlenbeck equations,*
- *when* $\gamma = \frac{1}{2}$, $\lambda_+ = 0$, *while* $\lambda_- = E_C - E_A$.

Case iii: $\sigma_{\pm}^2 = \frac{2}{9}(1 + \frac{c_{\mp}^2}{E_A - E_C} - c_{\mp})$ *and*

- *when* $\gamma > 1/2$, $\lambda_+ = \lambda_- = 0$,
- *when* $\gamma = 1/2$, $\lambda_+ = -\frac{(E_A-E_C)^2}{3\delta}(c_+ + \frac{E_C-E_B}{E_A-E_C})$, *while* $\lambda_- = \frac{(E_A-E_C)^2}{3\delta}(c_- + \frac{E_C-E_B}{E_A-E_C})$

Proof (some elements)

- Markov process on $\Omega_N = \{A, B, C\}^{\mathbb{T}_N}$ and generator $\mathscr{L}_N$ acting on $f : \Omega_N \to \mathbb{R}$ as

$$\mathscr{L}_N f(\eta) = N^{a} \sum_{x \in \mathbb{T}_N} c_x(\eta)[f(\eta^{x,x+1}) - f(\eta)], \qquad c_x(\eta) = \sum_{\alpha,\beta \in \{A,B,C\}} (1 + \frac{E_\alpha - E_\beta}{2N^{\gamma}})\xi_x^\alpha \xi_x^\beta$$

- Recall the density fluctuation field $\mathscr{Y}_t^{N,\alpha}(du) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{T}_N} [\xi_x^\alpha(\eta_t) - \varrho_\alpha]\, \delta_{\frac{x}{N}}(du)$
- and Dynkin's formula: for each $g : \Omega_N \to \mathbb{R}$ local

$$g(\eta_t) - g(\eta_0) - \int_0^t (\mathscr{L}_N + \partial_s) g(\eta_s) ds = \mathscr{M}_t^N(g)$$

  is a martingale with respect to the natural filtration of $g(\eta_t)$
- We take $g(\eta_t) = \mathscr{Y}_t^{N,\alpha}(f)$ with f smooth:

$$\mathscr{Y}_t^{N,\alpha}(f) - \mathscr{Y}_0^{N,\alpha}(f) - \int_0^t (\mathscr{L}_N + \partial_s) \mathscr{Y}_s^{N,\alpha}(f) ds = \mathscr{M}_t^{N,\alpha}(f)$$

Proof (some elements)

- Gradient system: $\mathscr{L}_N \bar{\xi}_x^\alpha = j_{x-1,x}^\alpha - j_{x,x+1}^\alpha$, where $\bar{\xi}_x^\alpha = \xi_x^\alpha - \varrho_\alpha$ and $j_{x,x+1}^\alpha$ is the infinitesimal (centered) current ($\alpha = A$)

$$j_{x,x+1}^A = \bar{\xi}_x^A - \bar{\xi}_{x+1}^A + \frac{E_C - E_A}{N^\gamma} \bar{\xi}_x^A \bar{\xi}_{x+1}^A - \frac{E_B - E_C}{2N^\gamma} (\bar{\xi}_x^A \bar{\xi}_{x+1}^B + \bar{\xi}_x^B \bar{\xi}_{x+1}^A) - \frac{E_B - E_A}{3N^\gamma} \bar{\xi}_x^A - \frac{E_B - E_C}{3N^\gamma} \bar{\xi}_x^B$$

- Dynkin's formula becomes

$$\mathscr{M}_t^{N,A}(f) = \mathscr{Y}_t^{N,A}(f) - \mathscr{Y}_0^{N,A}(f) - \int_0^t \mathscr{Y}_s^{N,A}(\Delta_N f) ds - \int_0^t \mathscr{B}^{AB}(\nabla_N f) ds - \int_0^t \mathscr{R}^{AB}(\nabla_N f) ds - \int_0^t \partial_s \mathscr{Y}_s^{N,A}(f) ds$$

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$$- \boxed{\frac{E_B - E_A}{3N^\gamma}\bar{\xi}_x^A - \frac{E_B - E_C}{3N^\gamma}\bar{\xi}_x^B}$$

- Dynkin's formula becomes

$$\mathscr{M}_t^{N,A}(f) = \mathscr{Y}_t^{N,A}(f) - \mathscr{Y}_0^{N,A}(f) - \int_0^t \boxed{\mathscr{Y}_s^{N,A}(\Delta_N f)}\, ds - \int_0^t \boxed{\mathscr{B}^{AB}(\nabla_N f)}\, ds$$
$$- \int_0^t \boxed{\mathscr{R}^{AB}(\nabla_N f)}\, ds - \int_0^t \partial_s \mathscr{Y}_s^{N,A}(f)\, ds$$

GOALS :

1. get rid of $\mathscr{R}^{AB}(\nabla_N f)$
2. express $\mathscr{B}^{AB}(\nabla_N f)$ as a function of the field

Proof (some elements)

SOLUTION:

1. We introduce a translation operator $T_{v_\alpha N^b t} f(\frac{x}{N}) = f\big(\frac{x+v_\alpha N^b t}{N}\big)$ in the field

$$\mathscr{Y}_t^{N,\alpha}(du) = \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} [\xi_x^\alpha(\eta_t) - \varrho_\alpha]\, T_{v_\alpha N^b t}\delta_{\frac{x}{N}}(du)$$

2. We look at linear combinations of $\mathscr{Y}_t^{N,A}$ and $\mathscr{Y}_t^{N,B}$

$$\mathscr{Z}_t^N(f) = a_1\mathscr{Y}_t^{N,A}(f) + a_2\mathscr{Y}_t^{N,B}(f)$$

and rewrite Dynkin's formula

$\Rightarrow$ Observe that $\partial_s\mathscr{Z}_s^N(f)$produces terms proportional to

$$a_1 v_A \frac{N^{b-1}}{\sqrt{N}} T_{v_A N^b t}\nabla f(\tfrac{x}{N})\bar{\xi}_x^A + a_2 v_B \frac{N^{b-1}}{\sqrt{N}} T_{v_B N^b t}\nabla f(\tfrac{x}{N})\bar{\xi}_x^B$$

$$\widetilde{\mathscr{M}}_t^{N,A}(f) = \mathscr{Z}_t^N(f) - \mathscr{Z}_0^N(f) - \int_0^t \mathscr{Z}_s^N(\Delta_N f)ds - \int_0^t \widetilde{\mathscr{B}}^{AB}(\nabla_N f)ds - \int_0^t \widetilde{\mathscr{R}}^{AB}(\nabla_N f)ds - \int_0^t \partial_s\mathscr{Z}_s^N(f)ds$$

Proof (some elements)

SOLUTION:

1. We introduce a translation operator $T_{v_\alpha N^b t} f(\frac{x}{N}) = f\big(\frac{x + v_\alpha N^b t}{N}\big)$ in the field

$$\mathcal{Y}_t^{N,\alpha}(du) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{T}_N} [\xi_x^\alpha(\eta_t) - \varrho_\alpha] \, T_{v_\alpha N^b t} \delta_{\frac{x}{N}}(du)$$

2. We look at linear combinations of $\mathcal{Y}_t^{N,A}$ and $\mathcal{Y}_t^{N,B}$

$$\mathcal{Z}_t^N(f) = a_1 \mathcal{Y}_t^{N,A}(f) + a_2 \mathcal{Y}_t^{N,B}(f)$$

and rewrite Dynkin's formula

$\Rightarrow$ Observe that $\partial_s \mathcal{Z}_s^N(f)$produces terms proportional to

$$a_1 v_A \frac{N^{b-1}}{\sqrt{N}} T_{v_A N^b t} \nabla f(\tfrac{x}{N}) \bar{\xi}_x^A + a_2 v_B \frac{N^{b-1}}{\sqrt{N}} T_{v_B N^b t} \nabla f(\tfrac{x}{N}) \bar{\xi}_x^B$$

$\Rightarrow$ When $a = b = 2$, if we choose $v_\alpha \sim \frac{1}{N^\gamma}$

$$\widetilde{\mathcal{M}}_t^{N,A}(f) = \mathcal{Z}_t^N(f) - \mathcal{Z}_0^N(f) - \int_0^t \mathcal{Z}_s^N(\Delta_N f) ds - \int_0^t \widetilde{\mathcal{B}}^{AB}(\nabla_N f) ds$$

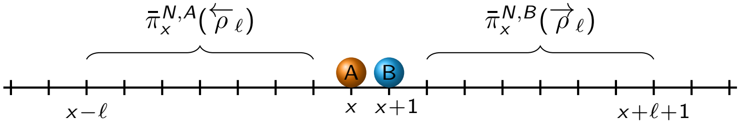
$$\cancel{- \int_0^t \widetilde{\mathcal{R}}^{AB}(\nabla_N f) ds - \int_0^t \partial_s \mathcal{Z}_s^N(f) ds}$$

Proof (some elements)

$$\widetilde{\mathscr{M}}_t^{N,A}(f) = \mathscr{Z}_t^N(f) - \mathscr{Z}_0^N(f) - \int_0^t \mathscr{Z}_s^N(\Delta_N f)ds - \int_0^t \widetilde{\mathscr{B}}^{AB}(\nabla_N f)ds$$

NEW GOAL : express $\widetilde{\mathscr{B}}^{AB}(\nabla_N f)$ as $G(\mathscr{Z}_s^N(\nabla_N f))$

SOLUTION : a "new" **second order Boltzmann–Gibbs principle** allows to replace local functionals of a conservative stochastic process by a nonlinear function of the conserved field



Theorem

There exists $C = C(\rho) > 0$ such that, for any $\varepsilon > 0$, $t > 0$, and $v \in \ell^2(\mathbb{Z})$

$$\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t ds \sum_{x\in\mathbb{Z}} v(x)\Big\{\bar{\xi}_x^A(s)\bar{\xi}_{x+1}^B(s) - \bar{\pi}_x^{N,A}(\overleftarrow{\rho}_{\varepsilon N})\bar{\pi}_x^{N,B}(\overrightarrow{\rho}_{\varepsilon N})\Big\}\Big)^2\Big] \le Ct\Big[\varepsilon + \frac{t}{N\varepsilon^2}\Big]\|v\|_{2,N}^2$$

Case i: $E_A = E_B = E$

1) $\mathscr{Z}_t^{N,-}(f) = \mathscr{Y}_t^{N,A}\big(T_{-\frac{E}{3}N^{3/2}t}f\big) + \mathscr{Y}_t^{N,B}\big(T_{-\frac{E}{3}N^{3/2}t}f\big)$ $\Rightarrow$ KPZ

2) $\mathscr{Z}_t^{N,+}(f) = \mathscr{Y}_t^{N,A}\big(T_{\frac{E}{3}N^{3/2}t}f\big) - \mathscr{Y}_t^{N,B}\big(T_{\frac{E}{3}N^{3/2}t}f\big)$ $\Rightarrow$ OU

.

Case i: $E_A = E_B = E$

1) $\mathscr{Z}_t^{N,-}(f) = \mathscr{Y}_t^{N,A}(T_{-\frac{E}{3}N^{3/2}t}f) + \mathscr{Y}_t^{N,B}(T_{-\frac{E}{3}N^{3/2}t}f) \quad \Rightarrow \quad \text{KPZ}$

The nonlinear term in Dynkin's formula is

$$\mathscr{B}_t^{N,-}(f) = -E\int_0^t ds \sum_{x\in\mathbb{T}_N} \nabla_N T_{-\frac{E}{3}N^{3/2}s} f\left(\tfrac{x}{N}\right)\left\{\bar{\xi}_x^A\bar{\xi}_{x+1}^A + \bar{\xi}_x^B\bar{\xi}_{x+1}^B + \bar{\xi}_x^A\bar{\xi}_{x+1}^B + \bar{\xi}_x^B\bar{\xi}_{x+1}^A\right\}$$

Applying the second order Boltzmann–Gibbs principle, we get

$$-E\int_0^t \mathrm{d}s \frac{1}{N}\sum_{x\in\mathbb{T}_N} \nabla_N T_{-\frac{E}{3}N^{3/2}s} f\left(\tfrac{x}{N}\right)\mathscr{Z}_s^{N,-}\left(\overleftarrow{i}_\varepsilon\left(\tfrac{x+\frac{E}{3}N^{3/2}s}{N}\right)\right)\mathscr{Z}_s^{N,-}\left(\overrightarrow{i}_\varepsilon\left(\tfrac{x+\frac{E}{3}N^{3/2}s}{N}\right)\right)$$

where $\overleftarrow{i}_\varepsilon(u)(v) = \frac{1}{\varepsilon}\mathbb{1}_{[u-\varepsilon,u)}(v)$ and $\overrightarrow{i}_\varepsilon(u)(v) = \frac{1}{\varepsilon}\mathbb{1}_{(u,u+\varepsilon]}(v)$.

In the limit $N\to\infty$ and $\varepsilon\to 0$ the martingale $\mathscr{M}_t^{N,-}(f)$ converges to the weak formulation of (SBE)

Case i: $E_A = E_B = E$

2) $\mathscr{Z}_t^{N,+}(f) = \mathscr{Y}_t^{N,A}\big(T_{\frac{E}{3}N^{3/2}t}f\big) - \mathscr{Y}_t^{N,B}\big(T_{\frac{E}{3}N^{3/2}t}f\big) \qquad \Rightarrow \qquad \text{OU}$

The quadratic term is

$$\mathscr{B}_t^{N,+}(f) = -\frac{E}{2}\int_0^t ds \sum_{x\in\mathbb{T}_N} \nabla_N T_{\frac{E}{3}N^{3/2}s} f\left(\tfrac{x}{N}\right) \Big\{(\bar{\xi}_x^A+\bar{\xi}_x^B)(\bar{\xi}_{x+1}^A-\bar{\xi}_{x+1}^B)+(\bar{\xi}_x^A-\bar{\xi}_x^B)(\bar{\xi}_{x+1}^A+\bar{\xi}_{x+1}^B)\Big\}$$

GOAL: show that $\mathscr{B}_t^{N,+}(f)$ vanishes in the $N \to \infty$ limit

By the Boltzmann–Gibbs principle

$$\mathscr{B}_t^{N,+}(f) = -\frac{E}{2}\int_0^t ds \frac{1}{N}\sum_{x\in\mathbb{T}_N} \nabla_N f\left(\tfrac{x}{N}\right) \Big\{\mathscr{Z}_s^{N,-}\Big(\overleftarrow{\rho_\varepsilon}\Big(\tfrac{x+\frac{2}{3}EN^{3/2}s}{N}\Big)\Big)\mathscr{Z}_s^{N,+}(\overrightarrow{\rho_\varepsilon}(\tfrac{x}{N}))$$

$$+\mathscr{Z}_s^{N,+}(\overleftarrow{\rho_\varepsilon}(\tfrac{x}{N}))\mathscr{Z}_s^{N,-}\Big(\overrightarrow{\rho_\varepsilon}\Big(\tfrac{x+\frac{2}{3}EN^{3/2}s}{N}\Big)\Big)\Big\}$$

Crossed fields

Theorem (Cannizzaro–Gonçalves–Misturini–O. '23)
Let v_1^N and v_2^N two diverging sequences of constants. Assume that:

- *for every $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$ such that $k_1 + k_2 \neq 0$*

$$\lim_{N\to\infty} \frac{1}{N} \left| k_1 v_1^N + k_2 v_2^N \right| = \infty \, ,$$

- *there exist $\alpha \in (0,1)$, and $C > 0$ such that for any $f_1, f_2 \in \mathcal{D}(\mathbb{T})$, uniformly in N, for all $0 \leq s \leq t \leq T$,*

$$\mathbb{E}\Big[\sup_{s\leq T} |\mathcal{X}_s^{N,+}(f_1)\mathcal{X}_s^{N,-}(f_2)| \Big] \leq C(\|\Delta f_1\|_2 \vee \|f_1\|_\infty)(\|\Delta f_2\|_2 \vee \|f_2\|_\infty) \, ,$$

$$\mathbb{E}\Big[|\mathcal{X}_t^{N,+}(f_1)\mathcal{X}_t^{N,-}(f_2) - \mathcal{X}_s^{N,+}(f_1)\mathcal{X}_s^{N,-}(f_2)| \Big] \leq C(t-s)^\alpha \|\Delta f_1\|_2 \|\Delta f_2\|_2 \, .$$

Then, for any $t \in [0, T]$, $f \in \mathcal{D}(\mathbb{T})$ and smooth and compactly supported φ_1, φ_2, we have

$$\lim_{N\to\infty} \mathbb{E}\left[\left| \int_0^t ds \frac{1}{N} \sum_{z\in\mathbb{T}_N} \nabla_N f\left(\tfrac{z}{N}\right) \mathcal{X}_s^{N,+}\left(\varphi_1\left(\tfrac{z+v_1^N s}{N}\right)\right) \mathcal{X}_s^{N,-}\left(\varphi_2\left(\tfrac{z+v_2^N s}{N}\right)\right) \right| \right] = 0$$

Remark This result can be generalised to an arbitrary number of fields

Case i: $E_A = E_B = E$

- Back to Dynkin's formula

$$\mathscr{M}_t^{N,+}(f) = \mathscr{Z}_t^{N,+}(f) - \mathscr{Z}_0^{N,+}(f) - \int_0^t ds \mathscr{Z}_s^{N,+}(\Delta_N f)$$
$$-\frac{E}{2}\int_0^t ds \frac{1}{N}\sum_{x\in\mathbb{T}_N} \nabla_N f\left(\tfrac{x}{N}\right)\Big\{\mathscr{Z}_s^{N,-}\Big(\overleftarrow{\rho_\varepsilon}\Big(\tfrac{x+\frac{2}{3}EN^{3/2}s}{N}\Big)\Big)\mathscr{Z}_s^{N,+}(\overrightarrow{\rho_\varepsilon}(\tfrac{x}{N}))$$
$$+\mathscr{Z}_s^{N,+}(\overleftarrow{\rho_\varepsilon}(\tfrac{x}{N}))\mathscr{Z}_s^{N,-}\Big(\overrightarrow{\rho_\varepsilon}\Big(\tfrac{x+\frac{2}{3}EN^{3/2}s}{N}\Big)\Big)\Big\}$$

- We apply the **Crossed fields theorem** with $\varphi_1(\cdot) = \overleftarrow{\rho_\varepsilon}(\cdot)$ and $\varphi_2(\cdot) = \overrightarrow{\rho_\varepsilon}(\cdot)$

⇒ The last term vanishes in $L^2(\mathbb{P}_{\nu_\rho})$ as $N \to \infty$

⇒ $\mathscr{M}^{N,+}(f)$ converges to the martingale associated to the (SBE) with $\lambda = 0$

Case i: $E_A = E_B = E$

- Back to Dynkin's formula

$$\mathscr{M}_t^{N,+}(f) = \mathscr{Z}_t^{N,+}(f) - \mathscr{Z}_0^{N,+}(f) - \int_0^t ds \mathscr{Z}_s^{N,+}(\Delta_N f)$$
$$- \frac{E}{2} \int_0^t ds \frac{1}{N} \sum_{x \in \mathbb{T}_N} \nabla_N f\left(\tfrac{x}{N}\right) \Big\{ \mathscr{Z}_s^{N,-}\Big(\overleftarrow{\rho_\varepsilon}\Big(\tfrac{x + \frac{2}{3} E N^{3/2} s}{N}\Big)\Big) \mathscr{Z}_s^{N,+}(\overrightarrow{\rho_\varepsilon}(\tfrac{x}{N}))$$
$$+ \mathscr{Z}_s^{N,+}(\overleftarrow{\rho_\varepsilon}(\tfrac{x}{N})) \mathscr{Z}_s^{N,-}\Big(\overrightarrow{\rho_\varepsilon}\Big(\tfrac{x + \frac{2}{3} E N^{3/2} s}{N}\Big)\Big) \Big\}$$

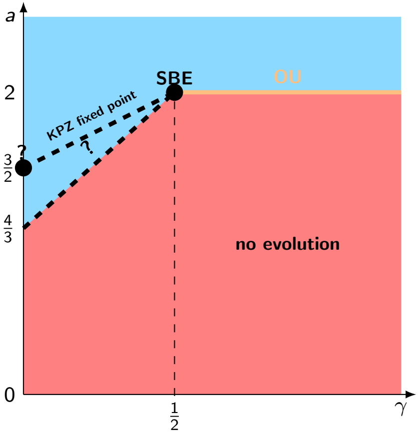
- We apply the **Crossed fields theorem** with $\varphi_1(\cdot) = \overleftarrow{\rho_\varepsilon}(\cdot)$ and $\varphi_2(\cdot) = \overrightarrow{\rho_\varepsilon}(\cdot)$

$\Rightarrow$ The last term vanishes in $L^2(\mathbb{P}_{\nu_\rho})$ as $N \to \infty$

$\Rightarrow$ $\mathscr{M}^{N,+}(f)$ converges to the martingale associated to the (SBE) with $\lambda = 0$

Remark Showing that the crossed terms vanish is equivalent to showing that non-diagonal terms of coupling matrices are negligeable

And then?



a
2
SBE
OU
KPZ fixed point
?
?
$\frac{3}{2}$
$\frac{4}{3}$
no evolution
0
$\frac{1}{2}$
γ

Ite, missa est

(Nunc est bibendum)