Asymptotic Algebraic Combinatorics I: lozenge tilings

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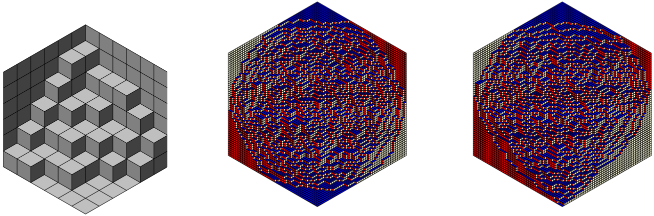
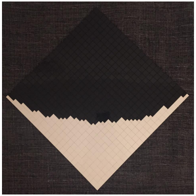
Dimers, Asymptotics and Algebraic combinatorics

Part 1. From plane partitions and symmetric functions to limit behavior of lozenge tilings... and back.



Part 2. Asymptotic Algebraic Combinatorics and Representation Theory: the quest for understanding structure constants (dimensions, Kostka, Littlewood-Richardson, Kronecker coefficients)



(©Dan Betea, at IHP Paris)

Integer and plane partitions

Integer partitions $\lambda \vdash n : \lambda = (\lambda_1, \ldots, \lambda_\ell)$, s.t
$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$, $|\lambda| := \lambda_1 + \lambda_2 + \cdots = n$

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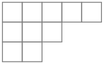
Young diagram of $\lambda = (5, 3, 2)$:



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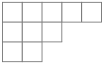
Young diagram of $\lambda = (5, 3, 2)$: 

$$\sum_{\lambda} q^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1}{1 - q^i}$$

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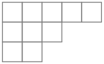
Young diagram of $\lambda = (5, 3, 2)$: 

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Plane partitions
$\pi : \mathbb{N}^2 \to \mathbb{Z}_{\geq 0}$, s.t.

$$\pi(i,j) \geq \pi(i+1,j), \pi(i,j+1) \qquad |\pi| := \sum_{i,j} \pi(i,j)$$

4	4	3	1	1	0	..
4	3	2	1	0	..	
2	2	1	0	..		
1	1	0	..			

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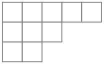
Young diagram of $\lambda = (5, 3, 2)$: 

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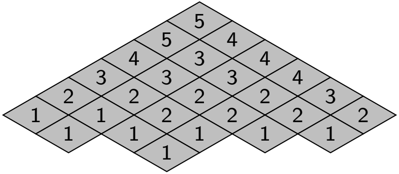
1	1	0	..			

MacMahon:

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{\infty} \frac{1}{(1 - q^i)^i}$$

Plane partitions and dimers

5	4	4	4	3	2
5	3	3	2	2	1
4	3	2	2	1	
3	2	2	1		
2	1	1	1		
1	1				



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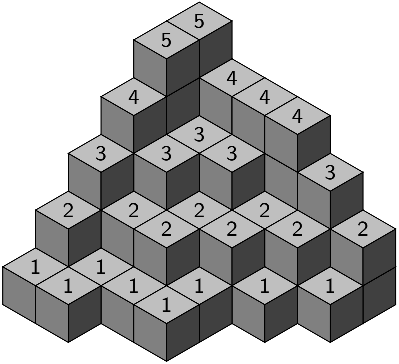

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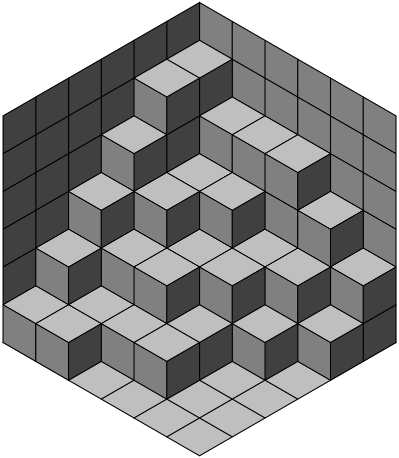

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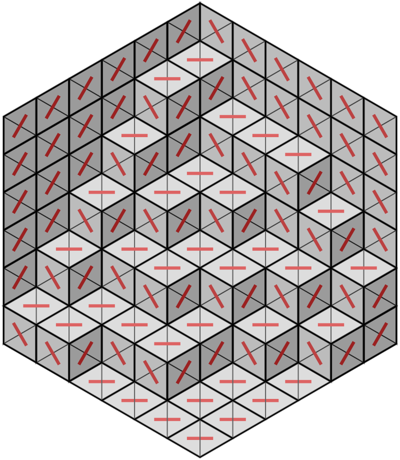

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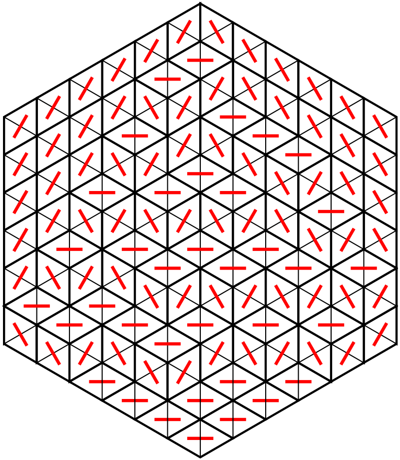

Bijections

Hillman-Grassl map Φ: Reverse Plane Partitions of shape λ to Arrays of shape λ:

$$\mathit{RRP}\ \ \pi = \begin{array}{ccc} 0&1&2\\ 1&1&3\\ 2&& \end{array} \to \begin{array}{ccc} 0&1&2\\ 1&1&3\\ 1&& \end{array} \to \begin{array}{ccc} 0&0&1\\ 0&0&3\\ 0&& \end{array} \to \begin{array}{ccc} 0&0&1\\ 0&0&2\\ 0&& \end{array} \to \begin{array}{ccc} 0&0&1\\ 0&0&1\\ 0&& \end{array}, \begin{array}{ccc} 0&0&0\\ 0&0&0\\ 0&& \end{array}$$

$$\begin{array}{ccc} 0&0&0\\ 0&0&0\\ 1&& \end{array} \to \begin{array}{ccc} 1&0&0\\ 0&0&0\\ 1&& \end{array} \to \begin{array}{ccc} 1&0&0\\ 0&0&1\\ 1&& \end{array} \to \begin{array}{ccc} 1&0&0\\ 0&0&2\\ 1&& \end{array} \to \begin{array}{ccc} 1&0&1\\ 0&0&2\\ 1&& \end{array} =: \mathit{Array}\ \ A = \Phi(P)$$

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$$Weight(\pi) = |\pi| = 0+1+2+1+1+3+2 = 10 =$$

$$= \sum_{i,j} A_{i,j} hook(i,j) = 1*5 + 1*2 + 2*1 + 1*1 =: weight(A)$$

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$$\sum_{\pi \in RPP(\lambda)} q^{|\pi|} = \sum_{A:Array(\lambda)} \prod_{(i,j)\in\lambda} q^{h(i,j)*A_{i,j}} = \prod_{(i,j)\in\lambda} \frac{1}{1-q^{h(i,j)}}$$

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$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & & \end{array} \to \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & & \end{array} \to \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & & \end{array} \to \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & & \end{array} \to \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & & \end{array} =: Array \quad A = \Phi(P)$$

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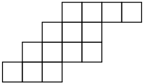
Corollary: MacMahon's formula

$$\sum_{\pi \in RPP(a^b)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{1}{1-q^{i+j-1}}$$

Skew (reverse) plane partitions

Skew shape λ/μ:  for $\lambda = (7,5,5,3)$, $\mu = (3,2,1)$

Skew RPP:

			0	1	1	3
		0	1	1		
	1	2	3	5		
1	2	2				

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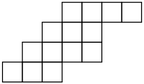
Skew shape λ/μ:  for $\lambda = (7,5,5,3)$, $\mu = (3,2,1)$

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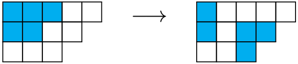
Excited diagrams:

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\longrightarrow$  $\}$



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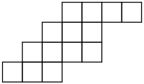
Skew shape λ/μ:  for $\lambda = (7,5,5,3)$, $\mu = (3,2,1)$

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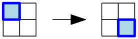
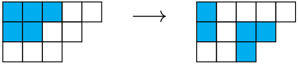
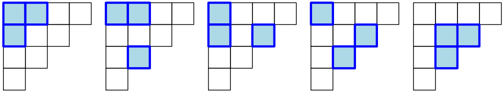
$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$

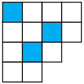


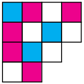


Skew (reverse) plane partitions

Skew RPPs ⇔ arrays with support *"pleasant diagrams"*:

$$PD(\lambda/\mu) := \{S \subset [\lambda] : S \subset [\lambda] \setminus D, \text{ for some } D \in \mathcal{E}(\lambda/\mu)\}$$

Excited: 

Pleasant boxes ■: 

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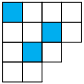
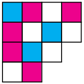
Excited: 

Pleasant boxes: 

Theorem (Morales-Pak-P)

The Hillman-Grassl map is a bijection between skew RPPs of shape λ/μ and arrays with support in the pleasant diagrams:

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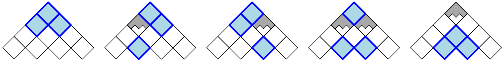

2^7 2^6 2^6 2^5 2^6

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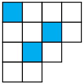
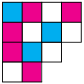
Excited: 

Pleasant boxes: 

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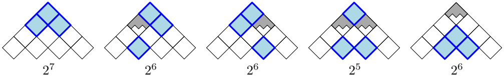

NEXT: add more variables

The ring of symmetric functions Λ

Λ_n = Formal power series in $x_1, x_2, \ldots$ of degree n, s.t. $f(x_1, x_2, \ldots) = f(x_{\sigma_1}, x_{\sigma_2}, \ldots)$ for all permutations σ.

$$\dim \Lambda_n = \#\{\lambda \vdash n\}$$

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Bases of Λ:
Monomial:

$$m_\lambda(x_1, x_2, \ldots) = \sum_{\sigma = perm(\lambda_1, \lambda_2, \ldots)} x_1^{\sigma_1} x_2^{\sigma_2} \cdots$$

E.g. $m_{(1,1)}(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_1x_3$, $m_{(2)}(x_1, x_2, \ldots) = x_1^2 + x_2^2 + \cdots$

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$$\begin{aligned} m_{(2,1,1)}(x_1, x_2, x_3, x_4, x_5) &= x_1^2x_2x_3 + x_2^2x_1x_3 + \cdots + x_5^2x_3x_4 \\ &= m_{(2,1,1)}(x_1, \ldots, x_4) + x_5 m_{(2,1)}(x_1 \ldots, x_4) + x_5^2 m_{(1,1)}(x_1, \ldots, x_4) \end{aligned}$$

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$$\begin{aligned} m_{(2,1,1)}(x_1, x_2, x_3, x_4, x_5) &= x_1^2x_2x_3 + x_2^2x_1x_3 + \cdots + x_5^2x_3x_4 \\ &= m_{(2,1,1)}(x_1, \ldots, x_4) + x_5 m_{(2,1)}(x_1 \ldots, x_4) + x_5^2 m_{(1,1)}(x_1, \ldots, x_4) \end{aligned}$$

Power sums:

$$p_n(x_1, \ldots) := x_1^n + x_2^n + \cdots \qquad p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots$$

$$\begin{aligned} p_2(x_1, \ldots) &= x_1^2 + x_2^2 + \cdots \\ p_{(2,1)}(x_1, \ldots) &= (x_1^2 + x_2^2 + \cdots)(x_1 + x_2 + \cdots) \\ &= m_3(x_1, \ldots) + m_{(2,1)}(x_1, \ldots) \end{aligned}$$

The ring of symmetric functions Λ

Homogeneous:

$$h_n(x_1,\ldots,x_N) := \sum_{a_1+\cdots+a_N=n} x_1^{a_1}x_2^{a_2}\cdots x_N^{a_N} = \sum_{\lambda\vdash n} m_\lambda(x_1,\ldots,x_N)$$

$$h_\lambda := h_{\lambda_1}h_{\lambda_2}\cdots$$

e.g. $h_n(\underbrace{1,\ldots,1}_{N}) = \binom{N+n-1}{n}$

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e.g. $h_n(\underbrace{1, \ldots, 1}_{N}) = \binom{N+n-1}{n}$

Elementary:

$$e_n(x_1, \ldots, x_N) := \sum_{1 \le i_1 < i_2 < \cdots < i_n \le N} x_{i_1} \cdots x_{i_n}$$

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$$

e.g. $e_n(\underbrace{1, \ldots, 1}_{N}) = \binom{N}{n}$

The Schur functions

Irreducible (polynomial) representations of the **General Linear group** $GL_N(\mathbb{C}) \to GL(V)$:

Weyl modules V_λ (aka $\mathcal{W}_\lambda$), indexed by highest weights λ, $\ell(\lambda) \leq N$.

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Characters or representations $\rho : G \to GL(V)$: $\chi_V(g) = \mathrm{Tr}(\rho(g))$
$\{\chi_V : V \in Irr(G)\}$ -orthonormal basis of central functions on G(const on conjugacy classes), $\chi_V \longleftrightarrow V$.

$$s_\lambda(x_1, \ldots, x_N) = \chi_{V_\lambda}\left(\begin{bmatrix} x_1 & 0 & \cdots \\ 0 & x_2 & \cdots \\ \vdots & \ddots & \cdots \end{bmatrix}\right)$$

Special cases:

$$s_{(n)} = h_n \qquad s_{(1^n)} = e_n$$

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$$s_{\underbrace{(k-1, k-2, \ldots, 1)}_{\delta_k}}(x_1, \ldots, x_k) = \frac{\det\left[x_i^{2(k-j)}\right]_{i,j=1}^k}{\prod_{i<j}(x_i - x_j)} = \prod_{i<j}(x_i + x_j)$$

Schur functions, continued

Jacobi-Trudi identity:

$$s_{\lambda_1,\ldots,\lambda_k} = \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+k-2} \\ \vdots & \ddots & h_{\lambda_i+k-j} & \vdots \end{bmatrix}_{i,j=1}^{k}$$

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Semi-Standard Young tableaux of shape λ :

$$s_{(2,2)}(x_1,x_2,x_3) = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2.$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}$$

MacMahon second time

SSYT shape $\lambda = (a^b)$ and entries $0, 1, 2, \ldots, b + c - 1$:

$$\begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&4\\\hline 2&3&3&4&5\\\hline 4&4&5&6&6\\\hline\end{array} - \begin{array}{|c|}\hline 0\\\hline 1\\\hline 2\\\hline\end{array} \to \begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&4\\\hline 1&2&2&3&4\\\hline 2&2&3&4&4\\\hline\end{array} = \text{ RPP entries } 0, 1, \ldots, c$$

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$$\sum_{\pi \in RPP(a \times b \times c)} q^{|\pi|} = q^{-\binom{b}{2}a} s_{a^b}(1, q, q^2, \ldots, q^{b+c-1})$$

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$$= q^{-\binom{b}{2}a} \frac{\det[q^{(b+c-i)(\lambda_j+b+c-j)}]_{i,j=1}^{b+c}}{\prod_{i<j}(q^{b+c-i} - q^{b+c-j})} = q^{-\binom{b}{2}a} \prod_{i<j} \frac{(q^{\lambda_i+b+c-i} - q^{\lambda_j+b+c-j})}{q^{b+c-i} - q^{b+c-j}} = \ldots$$

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$$\ldots = \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - q^{a+j+k-1}}{1 - q^{j+k-1}}$$

RSK and Cauchy

Robinson-Schensted-Knuth: $(P, Q) \longleftrightarrow A$, $col(A) = type(P)$, $row(A) = type(Q)$, P, Q SSYT, $sh(P) = sh(Q)$

RSK and Cauchy

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$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 3 & 0 \end{bmatrix} \to \begin{pmatrix} 2 & 3 & 3 & 1 & 1 & 2 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \end{pmatrix}$$

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$$\to \left(\begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & & \\ 3 & & & & \end{array}, \begin{array}{ccccc} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & & \\ 3 & & & & \end{array} \right)$$

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$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 3 & 0 \end{bmatrix} \to \begin{pmatrix} 2 & 3 & 3 & 1 & 1 & 2 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \end{pmatrix}$$

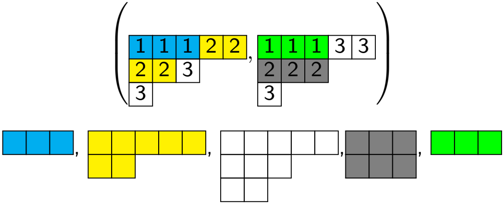
$$\to \left(\begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & & \\ 3 & & & & \end{array}, \begin{array}{ccccc} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & & \\ 3 & & & & \end{array} \right)$$

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_A \prod_{i,j} (x_i y_j)^{A_{i,j}} = \sum_{P,Q} x^{type(P)} y^{type(Q)} = \sum_\lambda s_\lambda(x) s_\lambda(y)$$

MacMahon again

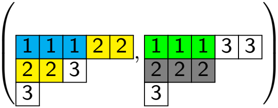
$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & & \\ 3 & & & & \end{array} , \begin{array}{ccccc} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & & \\ 3 & & & & \end{array} \right)$$

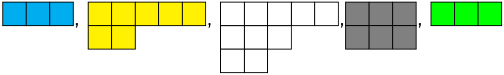
MacMahon again



$(3)\,,\quad (5,2)\,,\quad (5,3,2)\quad ,\,(3,3)\quad ,\,(3)$

MacMahon again

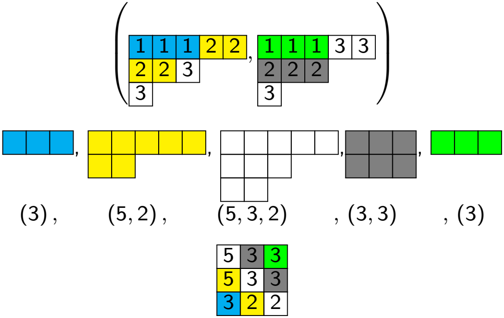




$(3)\,, \quad (5,2)\,, \quad (5,3,2)\,, \quad (3,3)\,, \quad (3)$

5	3	3
5	3	3
3	2	2

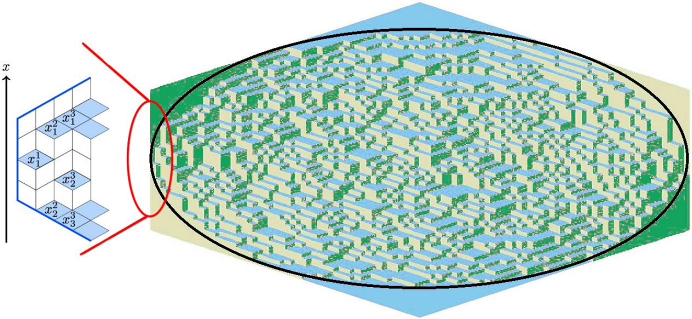
MacMahon again



$$\sum_{\pi \in RPP(a^b)} q^{|\pi|} = \sum_{\lambda} s_\lambda(1, q, \ldots, q^{a-1}) s_\lambda(q, q^2, \ldots, q^b) = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{1}{1 - q^{i+j-1}}$$

Classical questions: limit behavior

Question: Fix Ω in the plane and let *grid size* $\to 0$, what are the properties of *uniformly random* tilings of Ω?

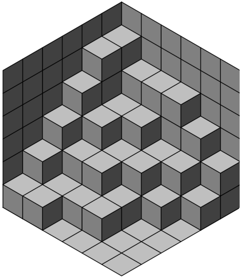


Behaviour near boundary (GUE), limit shapes (of the surface), frozen regions etc.
Central topic in Integrable Probability, Statistical Mechanics and Random Matrices.

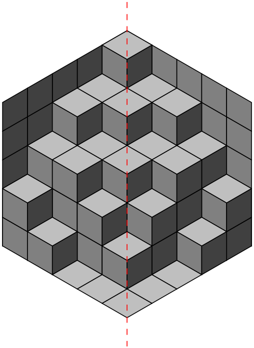
Unrestricted and symmetric lozenge tilings

Tilings of the hexagon $a \times b \times c \times a \times b \times c$, s.t.

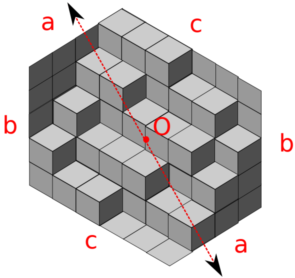
Unrestricted



Vertically symmetric



Centrally symmetric

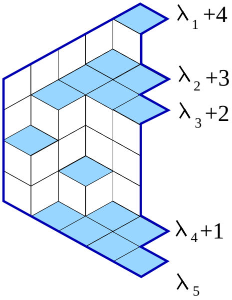


Limit behavior: fluctuations near the boundary (GUE), limit surface, CLT?

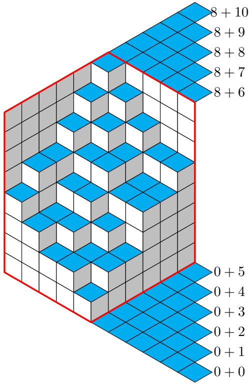
The Schur generating function: domain setup

Domain $\Omega_{\lambda(N)}$:
positions of the N horizontal lozenges on right boundary are:

$$\lambda_1(N) + N - 1 > \lambda_2(N) + N - 2 > \cdots > \lambda_N(N)$$

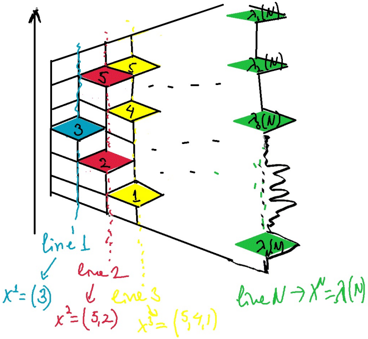


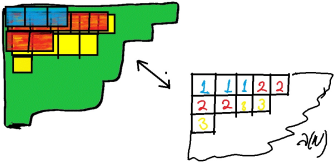
$\lambda(5) = (4, 3, 3, 0, 0)$



$\lambda = (\underbrace{a, \ldots, a}_{c}, \underbrace{0, \ldots, 0}_{b})$
$\leftrightarrow a \times b \times c$... hexagon.

Tilings probability: skew SSYTs



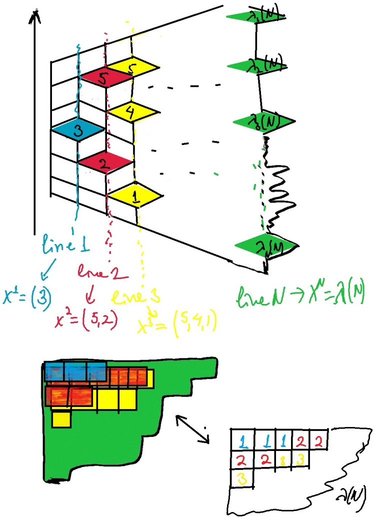


Lozenge tilings with right boundary $\lambda(N)$

$\Longleftrightarrow$

Semi-Standard Young Tableaux T of shape $\lambda(N)$ and entries $1, \ldots, N$.

Tilings probability: skew SSYTs



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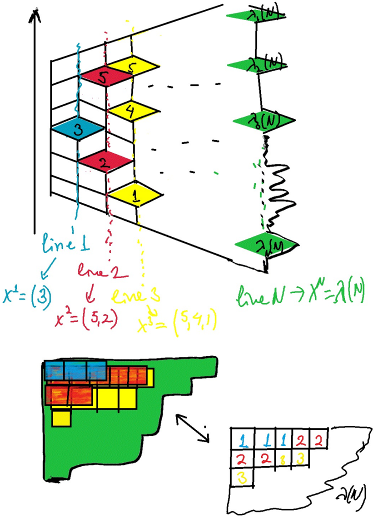
Tilings with horizontal lozenges on vertical line k at positions $x^k = (\eta_1, \ldots, \eta_k) = \eta$

$\Longleftrightarrow$

SSYTs T whose entries $1..k$ have shape η

Tilings probability: skew SSYTs



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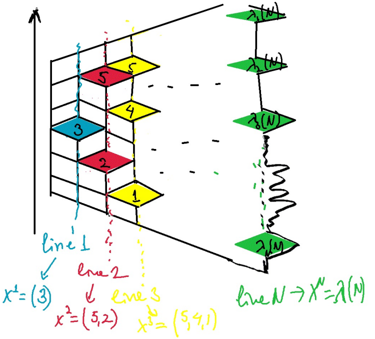
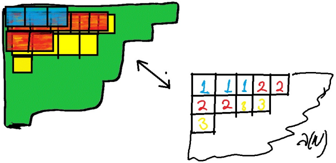
SSYTs T whose entries $1..k$ have shape η

$$\mathrm{Prob}\{x^k(\lambda) = \eta\} = \frac{s_\eta(1^k)s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$

Tilings probability: skew SSYTs





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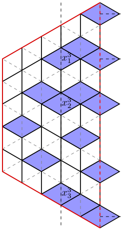
$$\mathrm{Prob}\{x^k(\lambda) = \eta\} = \frac{s_\eta(1^k)s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$

Proposition[Gorin-P'2013] For any variables $y_1, \ldots, y_k$, the **Schur Generating Function** of x^k is $S_\lambda(y_1, \ldots, y_k) :=$

$$\mathbb{E}\left(\frac{s_{x^k}(y_1, \ldots, y_k)}{s_{x^k}(\underbrace{1, \ldots, 1}_{k})}\right) = \frac{s_\lambda(y_1, \ldots, y_k, \overbrace{1, \ldots, 1}^{N-k})}{s_\lambda(\underbrace{1, \ldots, 1}_{N})}$$

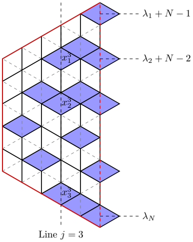
The explicit Schur Generating Functions[1]



$\mathcal{T}_n$ – set of tilings, $x^j(T)$ – horizontal lozenge positions on line j of $T \in \mathcal{T}_n$

[1] from [Gorin-P'2013], [P, 2014, 2015]

The explicit Schur Generating Functions[1]



$\mathcal{T}_n$ – set of tilings, $x^j(T)$ – horizontal lozenge positions on line j of $T \in \mathcal{T}_n$

$$\mathbb{E}\left[\frac{s_{x^k(T)}(y_1, \ldots, y_k)}{s_{x^k(T)}(\underbrace{1, \ldots, 1}_{k})} \;\middle|\; T \sim Unif(\mathcal{T}_n) \right]$$

$$= \sum_{\nu} \frac{s_\nu(y_1, \ldots, y_k)}{s_\nu(1^k)} \Pr(x^k(T) = \nu) = \ldots$$

[1] from [Gorin-P'2013], [P, 2014, 2015]

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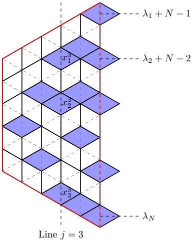

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$$= \sum_{\nu} \frac{s_\nu(y_1, \ldots, y_k)}{s_\nu(1^k)} \Pr(x^k(T) = \nu) = \ldots$$

- $= S_{\lambda(n)}(y_1, \ldots, y_k) = \frac{s_{\lambda(n)}(y_1, \ldots, y_k, 1^{n-k})}{s_{\lambda(n)}(1^n)}$ for $\mathcal{T}_n = \Omega_{\lambda(n)}$.
- $= \prod_i y_i^{m/2} \cdot \frac{\mathfrak{so}_{\left(\frac{m}{2}\right)^n}(y_1, \ldots, y_k, 1^{n-k})}{\mathfrak{so}_{\left(\frac{m}{2}\right)^n}(1^n)}$ for $\mathcal{T}_n$ – symmetric tilings of $n \times m \times n$....
- $= S_{\left(\frac{b}{2}\right)^{a/2}}(y_1, \ldots, y_k)^2$ for $\mathcal{T}_n$ – centrally symmetric tilings of $a \times b \times c$... hexagon.

[1] from [Gorin-P'2013], [P, 2014, 2015]

MGF asymptotics

Proposition (Gorin-P'2013)

$$\mathbb{E}_{\nu \sim \mathbb{GUE}_k} \left[\frac{s_{\nu-\delta_k}(y_1,\ldots,y_k)}{s_{\nu-\delta_k}(\underbrace{1,\ldots,1}_{k})} \right] = \exp\left(\frac{1}{2}(y_1^2+\cdots+y_k^2) \right),$$

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$$\mathbb{E}_{\text{tiling of }\Omega_\lambda(N)}\left(\frac{s_{x^k}(y_1,\ldots,y_k)}{s_{x^k}(\underbrace{1,\ldots,1}_{k})}\right) = \frac{s_{\lambda(N)}(y_1,\ldots,y_k,1^{N-k})}{s_{\lambda(N)}(1^N)} =: S_{\lambda(N)}(y_1,\ldots,y_k)$$

Proposition (Gorin-P'2013)

For any k real numbers $h_1,\ldots,h_k$ and $\lambda(N)/N\to f$ we have:

$$\lim_{N\to\infty} S_{\lambda(N)}\left(e^{\frac{h_1}{\sqrt{NS(f)}}},\ldots,e^{\frac{h_k}{\sqrt{NS(f)}}}\right)e^{\left(-\frac{E(f)}{\sqrt{NS(f)}}\sum_{i=1}^k h_i\right)} = \exp\left(\frac{1}{2}\sum_{i=1}^k h_i^2\right).$$

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Theorem (Gorin-P'2013)

Let $\Upsilon^k_{\lambda(N)}=\{x^k,x^{k-1},\ldots\}$ –collection of positions of the horizontal lozenges on lines $k,k-1,\ldots,1$ of tiling from $\Omega_{\lambda(N)}$, then

$$\frac{\Upsilon^k_{\lambda(N)}-NE(f)}{\sqrt{NS(f)}}\to\mathbb{GUE}_k\ \textit{(GUE-corners process of rank } k\textit{)}.$$

Asymptotics of normalized Schur functions

$$S_{\lambda(N)}(x_1,\ldots,x_k) := \frac{s_{\lambda(N)}(x_1,\ldots,x_k,\overbrace{1,\ldots,1}^{N-k})}{s_{\lambda(N)}(\underbrace{1,\ldots,1}_{N})}$$

Theorem [Gorin-P'2013] For every partition λ and any $x \in \mathbb{C} \setminus \{0,1\}$ we have

$$S_\lambda(x;N,1) = \frac{(N-1)!}{(x-1)^{N-1}}\frac{1}{2\pi\mathbf{i}}\oint_C \frac{x^z}{\prod_{i=1}^N(z-(\lambda_i+N-i))}dz,$$

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Theorem[Gorin-P'2013] If $\frac{\lambda_i(N)}{N} \to f\left(\frac{i}{N}\right)$ [...], for all fixed $y \neq 0$:

$$\lim_{N\to\infty}\frac{1}{N}\ln S_{\lambda(N)}(e^y;N,1) = yw_0 - \mathcal{F}(w_0) - 1 - \ln(e^y-1),$$

where $\mathcal{F}(w;f) = \int_0^1 \ln(w-f(t)-1+t)dt$, w_0 – root of $\frac{\partial}{\partial w}\mathcal{F}(w;f) = y$.
If $\frac{\lambda_i(N)}{N} \to f\left(\frac{i}{N}\right)$ [...], for any fixed $h \in \mathbb{R}$:

$$S_{\lambda(N)}(e^{h/\sqrt{N}};N,1) = \exp\left(\sqrt{N}E(f)h + \frac{1}{2}S(f)h^2 + o(1)\right),$$

where $E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 (f(t)-t+1/2)^2dt - 1/6 - E(f)^2.$

Asymptotics of normalized Schur functions

$$S_{\lambda(N)}(x_1,\ldots,x_k) := \frac{s_{\lambda(N)}(x_1,\ldots,x_k,\overbrace{1,\ldots,1}^{N-k})}{s_{\lambda(N)}(\underbrace{1,\ldots,1}_{N})}$$

$$S_\lambda(x_1,\ldots,x_k;N) = \prod_{i=1}^{k}\frac{(N-i)!}{(N-1)!(x_i-1)^{N-k}} \times \frac{\det\left[\left(x_i\frac{\partial}{\partial x_i}\right)^{j-1}\right]_{i,j=1}^{k}}{\Delta(x_1,\ldots,x_k)}\prod_{j=1}^{k} S_\lambda(x_j;N,1)(x_j-1)$$

If $\quad \dfrac{\ln\left(S_{\lambda(N)}(x;N,1)\right)}{N} \to \Psi(x)$ unif. on a compact $M \subset \mathbb{C}$. Then for any k

$$\lim_{N\to\infty}\frac{\ln\left(S_{\lambda(N)}(x_1,\ldots,x_k;N,1)\right)}{N} = \Psi(x_1)+\cdots+\Psi(x_k)$$

uniformly on M^k.

More informally, under various regimes of convergence for $\lambda(N)$ and $x_1,\ldots,x_k$ we have

$$S_{\lambda(N)}(x_1,\ldots,x_k) \sim S_{\lambda(N)}(x_1)\cdots S_{\lambda(N)}(x_k).$$

Asymptotics of normalized Schur functions

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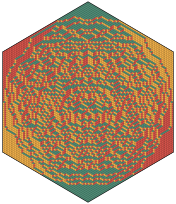
More normalized Schur function asymptotics: Novak, Petrov, Mkrtchyan, Zh. Li

Limit surface for symmetric tilings



Theorem (P, 2014)

Let $n, m \in \mathbb{Z}$, such that $m/n \to a$ as $n \to \infty$, where $a \in (0, +\infty)$. Let $H_n(u,v)$ – height function of a symmetric tiling of $n \times m \times n\ldots$ hexagon,i.e.

$$H_n(u,v) = \frac{1}{n} y^{\lfloor nu \rfloor}_{\lfloor nv \rfloor} - v.$$

For all $1 \geq u \geq v \geq 0$, as $n \to \infty$: $H_n(u,v)$ converges unif. in prob. to a deterministic function $L(u,v)$ ("the limit surface").

For any fixed $u \in (0,1)$, $L(u,v)$ is the distribution function of the measure $\mathbf{m}$, given by its moments:

$$\int_{\mathbb{R}} t^r \mathbf{m}(dt) = \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{1}{(\ell+1)!} u^{-r+\ell} \frac{\partial^\ell}{\partial z^\ell} z^p \Phi'_a(z)^{p-\ell} \Big|_{z=1},$$

where $\Phi_a(e^y) = y\frac{a}{2} + 2\phi(y;a) - 2$ and...

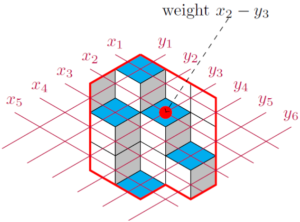
$$h(y) = \frac{1}{4}\left((e^y+1) + \sqrt{(e^y+1)^2 + 4(a^2+a)(e^y-1)^2} \right)$$

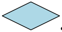
$$\phi(y;a) = (\frac{a}{2}+1)\ln\left(h(y) - (\frac{a}{2}+1)(e^y-1)\right) - (\frac{a}{2}+\frac{1}{2})\ln\left(h(y) - (\frac{a}{2}+\frac{1}{2})(e^y-1)\right)$$
$$+ \frac{a}{2}\ln\left(h(y) + \frac{a}{2}(e^y-1)\right) - (\frac{a}{2}-\frac{1}{2})\ln\left(h(y) + (\frac{a}{2}-\frac{1}{2})(e^y-1)\right)$$

Theorem (P, 2015)

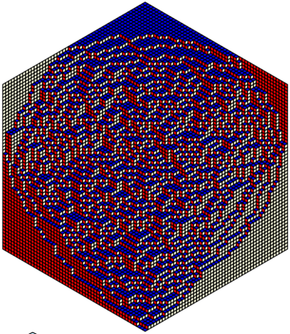
The scaled height function $H_n(u,v)$ of a centrally symmetric tiling of an $a \times b \times c\ldots$ hexagon converges uniformly in probability to a deterministic function $L(u,v)$ – the limit surface, as $n \to \infty$, where $n = \frac{a+c}{2}$ and $a/n, b/n$ – approx constant.
The limit surface coincides with the limit surface for the uniformly random tilings of the hexagon (without symmetry constraints).

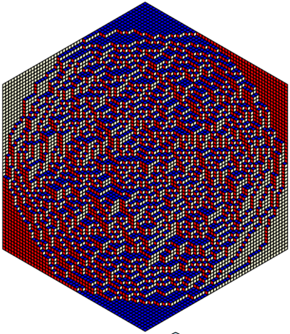
Multivariate local weights

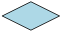


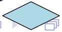
Total weight $= \prod$  at (i,j) $(x_i - y_j)$

$$(x_1 - y_1)(x_2 - y_3)(x_3 - y_5)(x_3 - y_2)(x_5 - y_5).$$





 at $(i,j) = 2N - (i+j)$

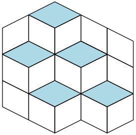
 $= 1$

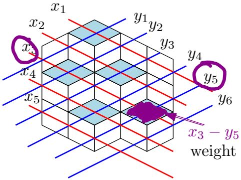
Lozenge tilings with multivariate weights

$\Omega_{\mu,d}$: Plane partitions with base μ, height d

weights of horizontal lozenges $= x_i - y_j$

3	2	1
2	1	





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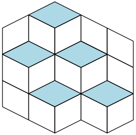
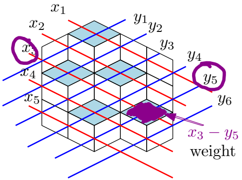




Theorem (Morales-Pak-P)

Consider tilings with base μ and height d, we have that

$$\sum_{T \in \Omega_{\mu,d}} \prod_{(i,j) \in T} (x_i - y_j) = \det[A_{i,j}(\mu,d)]_{i,j=1}^{d+\ell(\mu)},$$

where

$$A_{i,j}(\mu,d) := \begin{cases} \frac{(x_i - y_1)\cdots(x_i - y_{d+\ell(\mu)-j})}{(x_i - x_{i+1})\cdots(x_i - x_{d+\ell(\mu)})}, & \text{when } j = \ell(\mu)+1,\ldots,\ell(\mu)+d, \\ \frac{(x_i - y_1)\cdots(x_i - y_{\mu_j+d})}{(x_i - x_{i+1})\cdots(x_i - x_{d+j})}, & \text{when } j = i-d,\ldots,\ell(\mu), \\ 0, & \text{when } j < i-d. \end{cases}$$

Corollary (Krattenthaler, Stanley etc)

Consider the set $PP(\mu, d)$ of plane partitions of base μ and entries less than or equal to d. Then their volume generating function is given by the following determinantal formula

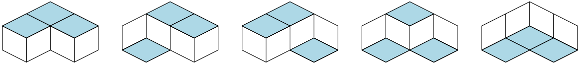
$$\sum_{P \in PP(\mu,d)} q^{|P|} = q^{\sum_r r\mu_r} \det[C_{i,j}]_{i,j=1}^{\ell+d},$$

where

$$C_{i,j} = \begin{cases} \frac{(-1)^{d+\ell-i} q^{(d-i)(d+\ell-j) - \frac{(d-i+\ell)(d-i-\ell-1)}{2}}}{(q;q)_{d+\ell-i}}, & \text{when } j = \ell+1, \dots, \ell+d, \\ \frac{(-1)^{d+j-i} q^{(d-i)(\mu_j+d) - \frac{(d+j-i)(d-i-j-1)}{2}}}{(q;q)_{d+j-i}}, & \text{when } j = i-d, \dots, \ell, \\ 0, & \text{when } j < i-d, \end{cases}$$

where $(q;q)_m = (1-q)\cdots(1-q^m)$ is the $q-$Pochhammer symbol.

E.g. $\mu = (2,1)$, $d = 1$:



$$\sum_{P \in PP((2,1),1)} q^{|P|} = q^0 + q^1 + 2q^2 + q^3$$

Theorem (Morales-Pak-P)

Tilings of the $a \times b \times c \times a \times b \times c$ ($\mu = a \times b$, $d = c$) hexagon with horizontal lozenges weights $x_i - y_j$ The partition function is given by

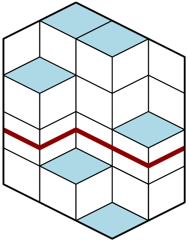
$$Z(a,b,c) := \sum_{T \in \Omega_{a,b,c}} \prod_{(i,j)\in T} (x_i - y_j) = \det \left[\begin{cases} \frac{(x_i-y_1)\cdots(x_i-y_{c+a-j})}{(x_i-x_{i+1})\cdots(x_i-x_{c+a})} & \text{if } j > a \\ \frac{(x_i-y_1)\cdots(x_i-y_{b+c})}{(x_i-x_{i+1})\ldots(x_i-x_{c+j})} & \text{if } j = i-c,\ldots,a \\ 0, & j < i-c \end{cases} \right]_{i,j=1}^{a+c}$$

Consider a path $P(d_1, \ldots)$ consisting of vertical lozenges passing through the points (i, d_i)

The probability that such path exists is given by

$$\mathrm{Prob(path)} = \frac{\det[A_{i,j}(\mu, d)] \det[\bar{A}_{i,j}(\mu^*, c-d-1)]}{Z}$$

where $d := d_1$, $\ell(\mu) = b$, $\mu_1 = a$ and diagonals$(\mu) = (d_1 - d, d_2 - d, \ldots)$, and $\mu^* = a \times b \setminus \mu$. Matrix $\bar{A}$ $x_i \to x_{a+c+1-i}$ and $y_j \to y_{b+c+1-j}$.

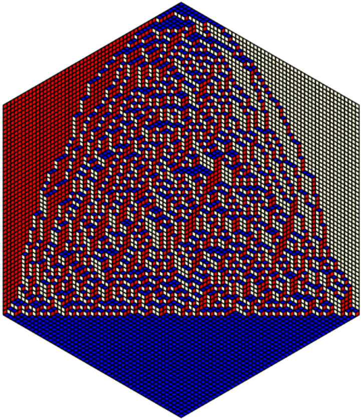


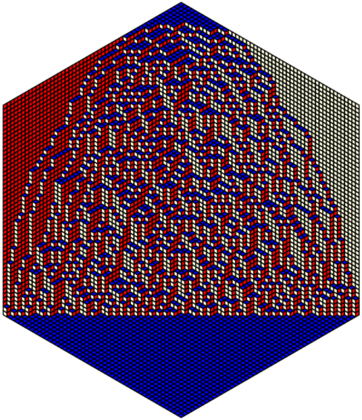
$\mu = 31$

$\mu^* = 20$

Simulation 2: base = δ_n

Weights: "hook" weights $(4n - i - j)$ versus uniform (i.e. 1).





Standard Young Tableaux

Basis for $\mathbb{S}_\lambda$ given by **SYTs** of shape λ:
$T : \lambda \xrightarrow{\sim} \{1, \ldots, n\}$ and $T_{i,j} < T_{i,j+1}, T_{i+1,j}$

$T = \wedge$

1	3	4	7	10
2	5	8		
6	9			

Standard Young Tableaux

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$$T = \wedge \begin{array}{|c|c|c|c|c|} \multicolumn{5}{c}{\quad\quad <} \\ \hline 1 & 3 & 4 & 7 & 10 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 6 & 9 \\ \cline{1-2} \end{array}$$

$\lambda = (3, 2, 1)$:

$$\begin{array}{ccc} 1&2&3 \\ 4&5 \\ 6 \end{array} \quad \begin{array}{ccc} 1&2&3 \\ 4&6 \\ 5 \end{array} \quad \begin{array}{ccc} 1&2&4 \\ 3&5 \\ 6 \end{array} \quad \begin{array}{ccc} 1&2&4 \\ 3&6 \\ 5 \end{array} \quad \begin{array}{ccc} 1&2&5 \\ 3&4 \\ 6 \end{array} \quad \begin{array}{ccc} 1&2&5 \\ 3&6 \\ 4 \end{array} \quad \begin{array}{ccc} 1&2&6 \\ 3&4 \\ 5 \end{array} \quad \begin{array}{ccc} 1&2&6 \\ 3&5 \\ 4 \end{array} \; + \; \textit{all transposed}$$

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Hook-length formula [Frame-Robinson-Thrall]:

$$f^\lambda := \#\{\text{SYTs of shape } \lambda\} = \frac{|\lambda|!}{\prod_{u \in \lambda} h_u} = \frac{6!}{5 * 3 * 3 * 1 * 1 * 1} = 16$$

Hook length of box $u = (i, j) \in \lambda$: $h_u = \lambda_i - j + \lambda'_j - i + 1 = \#\blacksquare \in$ 

Counting skew SYTs: formulas

Outer shape λ, inner – μ, e.g. for $\lambda = (5, 4, 4, 2, 1), \mu = (2, 2, 1)$:

Skew SYT:

		2	3	6
		5	8	
	1	7	10	
4	9			
11				

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Skew SYT:

		2	3	6
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	1	7	10	
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11				

Jacobi-Trudi[Feit 1953]:

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)} .$$

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Jacobi-Trudi[Feit 1953]:

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)}.$$

No product formula, e.g.

$\lambda/\mu = \delta_{n+2}/\delta_n$:

			5	6
		1	9	
	2	7		
3	4			
8				

$\leftrightarrow \quad 8 > 3 < 4 > 2 < 7 > 1 < 9 > 5 < 6$

$f^{\delta_{n+2}/\delta_n} = E_{2n+1}$:

$$1 + E_1 x + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + E_4 \frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

Euler numbers: $2, 5, 16, 61 \ldots$.

Hook-Length formula for skew shapes

Theorem (Naruse-Ikeda)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

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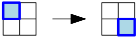
Excited diagrams:

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$

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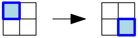
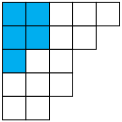
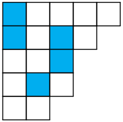
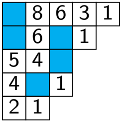
$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$

 $\longrightarrow$ 

Hook lengths inside λ: 

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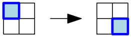
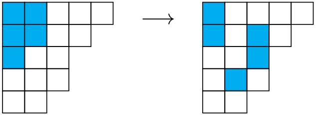
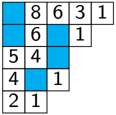
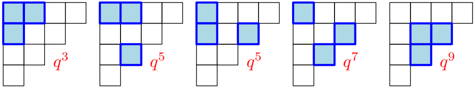
$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$

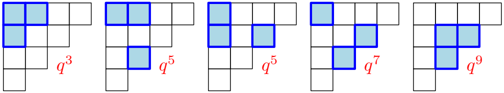


Hook lengths inside λ: 



$$f^{(4321/21)} = 7!\left(\frac{1}{1^4\cdot 3^3} + \frac{1}{1^3\cdot 3^3\cdot 5} + \frac{1}{1^3\cdot 3^3\cdot 5} + \frac{1}{1^2\cdot 3^3\cdot 5^2} + \frac{1}{1^2\cdot 3^2\cdot 5^2\cdot 7}\right) = 61$$

Hook-Length formula for skew shapes



$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in SSYT(4321/21)} q^{|T|} = \frac{q^3}{(1-q)^4(1-q^3)^3} + 2 \times \frac{q^5}{(1-q)^3(1-q^3)^3(1-q^5)} + \cdots$$

Theorem (Morales-Pak-P)

For skew SSYTs, we have that

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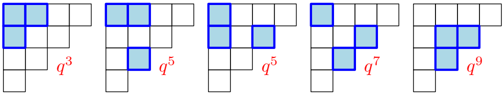
Hook-Length formula for skew shapes



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$$s_{(3,2)/(1)}(1,q,q^2,\cdots) = \underset{\begin{array}{ccc} & 0 & 0 \\ 0 & 1 & \end{array}}{q^{0+0+0+1}} + \underset{\begin{array}{ccc} & 0 & 1 \\ 0 & 1 & \end{array}}{q^{0+1+0+1}} + \cdots + \underset{\begin{array}{ccc} & 1 & 3 \\ 0 & 3 & \end{array}}{q^{1+3+0+3}} + \underset{\begin{array}{ccc} & 1 & 1 \\ 2 & 3 & \end{array}}{q^{1+1+2+3}} + \cdots$$

Proofs of NHLF

- Equivaraint Schubert Calculus [Naruse, generalized in MPP1] via Schubert class localization formulas at Grassmannian permutations, i.e. certain evaluation of Schubert polynomials = Factorial Schur functions.

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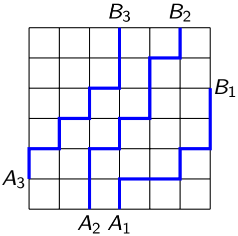
Lattice paths



Non-Intersecting Lattice Paths (NILP):

$(P_1, P_2, \ldots)$
$P_1 : A_1 \to B_1;\ P_2 : A_2 \to B_2;\ \ldots$

Theorem[Karlin–McGregor–Lindström–Gessel–Viennot]
(Number of) Nonintersecting Lattice Paths:

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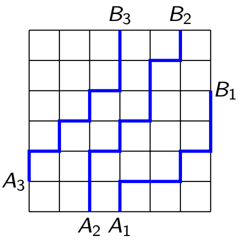

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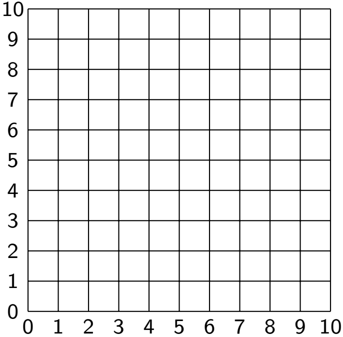
Proof: Sign reversing involution on intersecting pairs
$(A_{i_1} \to B_{j_1}, A_{i_2} \to B_{j_2}) \leftrightarrow (A_{i_1} \to B_{j_2}, A_{i_2} \to B_{j_1})$

Non-intersecting lattice paths



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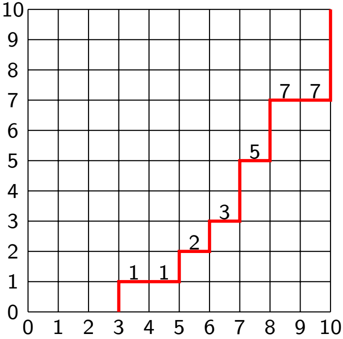

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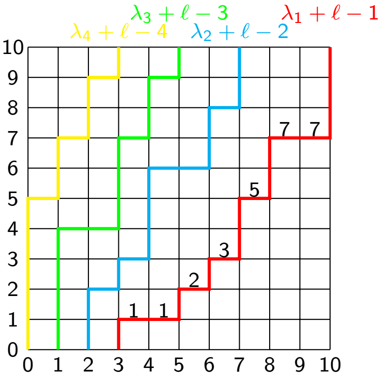

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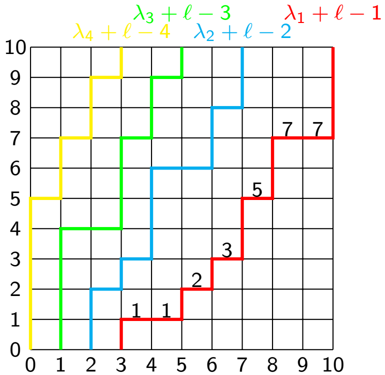

$NILP((\ell - i, 1) \to (\lambda_i + \ell - i, N)\ , i = 1 \to \ell)$

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$SSYT(\lambda; N)$

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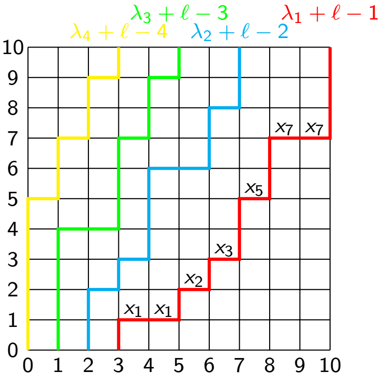

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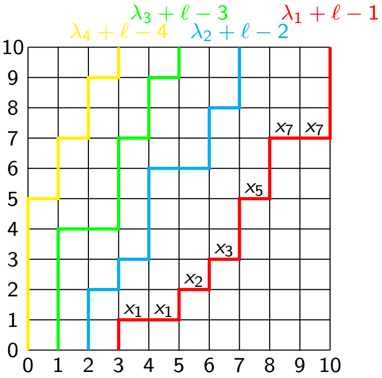

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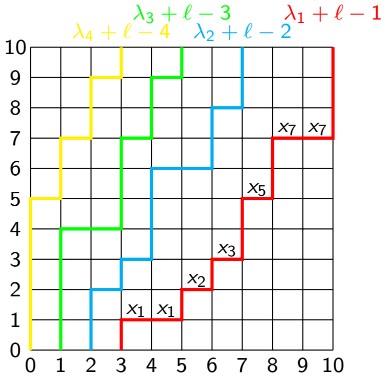

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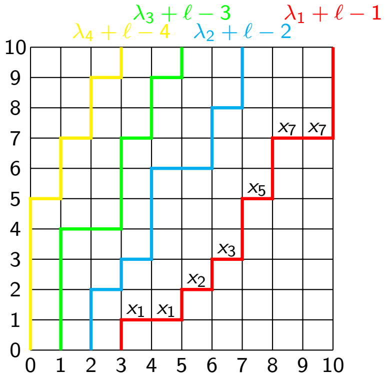

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$$A_i = (\ell - i, 1), B_i = (\lambda + \ell - i, N)$$

Jacobi-Trudi identity:

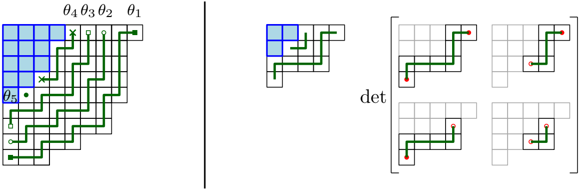
$$s_\lambda(x) = \sum_{P_1,\ldots,P_\ell : NILP(\mathbf{A}\to\mathbf{B})} \prod_i W(P_i) = \det[\sum_{P:A_i \to B_j} W(P)]_{i,j=1}^{\ell} = \det[h_{\lambda_i - i + j}]_{i,j=1}^{\ell}$$

NILP proof of NHLF

Theorem[Lascoux-Pragacz, Hamel-Goulden] If $(\theta_1, \ldots, \theta_k)$ is a Lascoux–Pragacz decomposition (i.e. maximal outer border strip decomposition) of λ/μ, then

$$s_{\lambda/\mu} = \det \left[s_{\theta_i \# \theta_j} \right]_{i,j=1}^{k}.$$

where $s_\varnothing = 1$ and $s_{\theta_i \# \theta_j} = 0$ if the $\theta_i \# \theta_j$ is undefined.
θ_1 – border strip following the inner border of λ;
θ_i – inner border of $\lambda \setminus (\theta_1 \cup \cdots \cup \theta_{i-1})$ etc until μ is hit,
then – border strips from each connected part etc.
Ordering: corners.
Strip $\theta_i \# \theta_j :=$ shape of θ_1 between the diagonals of the endpoints of θ_i and θ_j.



NHLF for border strips

Lemma (MPP)

For a border strip $\theta = \lambda/\mu$ with end points (a,b) and (c,d) we have

$$s_\theta(1,q,q^2,\ldots,) = \sum_{\substack{\gamma:(a,b)\to(c,d),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j - i}}{1-q^{h(i,j)}}.$$

s  $(1,q,q^2,\ldots) = \dfrac{q^3}{(1-q^2)(1-q^1)(1-q^3)(1-q^1)(1-q^2)}$ $\gamma=(3,1),(3,2),(2,2),(2,3),(1,3)$:  $+ \dfrac{q^4}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$ 

$+ \dfrac{q^1}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$  $+ \dfrac{q^7}{(1-q)^2(1-q^3)(1-q^4)^2}$  $+ \dfrac{q^6}{(1-q)^2(1-q^5)(1-q^4)^2}$ 

Proofs: induction on $|\lambda/\mu|$, or [multivariate] Chevalley formula for factorial Schurs.

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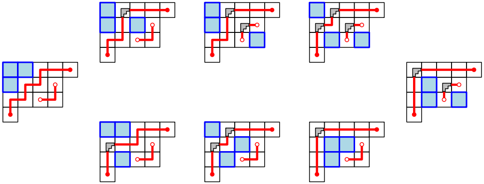

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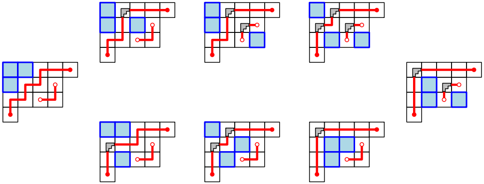
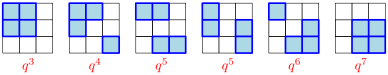


$$s_{\lambda/\mu} \underset{\text{Lascoux-Pragacz}}{=} \det\left[s_{\theta_i \# \theta_j} \right]_{i,j=1}^k \underset{s_{\text{Border Strip}}}{=} \det\left[\sum_{\gamma:(a_i,b_i)\to(c_j,d_j)} \prod_{u\in\gamma} \frac{q^{\cdot\cdot}}{1-q^{h_u}} \right]$$

$$\underset{Lindstrom-Gessel-Viennot}{=} \sum_{NILP:\gamma_1,\dots} \prod_{u\in\gamma_1\cup\dots} \frac{q^{\cdot\cdot}}{1-q^{h_u}} \underset{\mathcal{E}(\lambda/\mu)=NILP}{=} \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{u\in D} \frac{q^{\cdots}}{1-q^{h_u}}$$

Factorial Schur functions, multivariate lozenge tilings

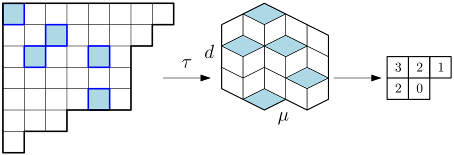


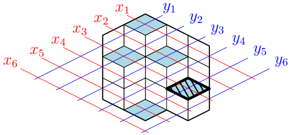
Theorem (Ikeda-Naruse, also cor to Kreiman+Knutson-Tao)

Let $\mu \subset \lambda \subset d \times (n-d)$*. Let* $v(n-d+1-i) = \lambda_i + (n-d+1-i)$ *and* $v(j) = d + j - \lambda'_j$*. Then*

$$s^{(d)}_{\mu}(y_{v(1)}, \ldots, y_{v(d)} | y_1, \ldots, y_{n-1}) = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} (y_{v(d-i+1)} - y_{v(d+j)})$$

$$\Longrightarrow = \frac{\det\left[(y_{v(j)} - y_1)\cdots(y_{v(j)} - y_{\mu_i+d-i})\right]_{i,j=1}^{d}}{\prod_{1 \le i < j \le d}(y_{v(i)} - y_{v(j)})},$$

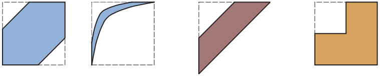




Applications of NHLF

- Asymptotics of $f^{\lambda/\mu}$:

$$\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \frac{1}{2} n \log n$$



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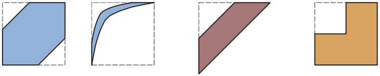

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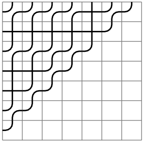

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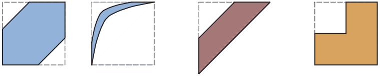

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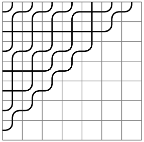
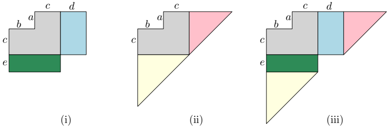


- Explicit product formulas for some $f^{\lambda/\mu}$.



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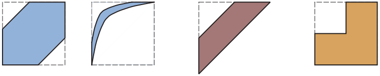
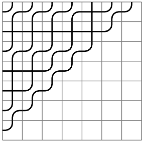
- Asymptotics of $f^{\lambda/\mu}$:

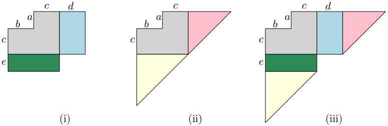
$$\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \frac{1}{2} n \log n$$



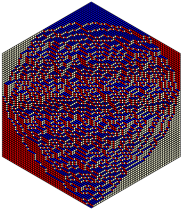
- Principle evaluations of Schubert polynomials (pipe dreams) and asymptotics.



- Explicit product formulas for some $f^{\lambda/\mu}$.



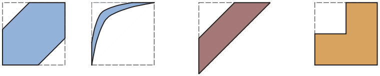
- Weighted lozenge tilings.



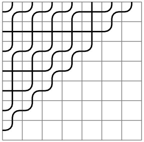
Applications of NHLF

- Asymptotics of $f^{\lambda/\mu}$:

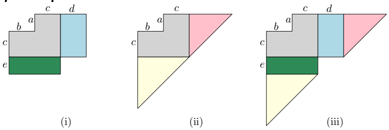
$$\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \frac{1}{2} n \log n$$



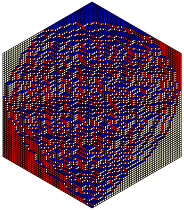
- Principle evaluations of Schubert polynomials (pipe dreams) and asymptotics.



- Explicit product formulas for some $f^{\lambda/\mu}$.

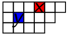


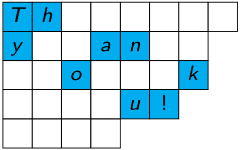
- Weighted lozenge tilings.



- Sorting probabilities for Young diagrams.

$$|\Pr[x < y] - \Pr[y < x]| \to 0$$

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