Asymptotic Algebraic Combinatorics II: structure constants

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Symmetric group S_n
Irreps $\mathbb{S}_\lambda$, $\lambda \vdash n$

$$Tr_{\mathbb{S}_\lambda}[\pi] = \chi^\lambda(\pi)$$

Standard Young Tableaux (SYT)

1	3	4	7	9
2	6	10		
5	8			

General linear group GL_N
V_λ, $\ell(\lambda) \leq N$

$$Tr_{V_\lambda}(diag(x_1, \ldots)) = s_\lambda(x_1, x_2, \ldots)$$

Semi-Standard Young Tableaux(SSYT)

1	1	1	2	3
2	2	3		
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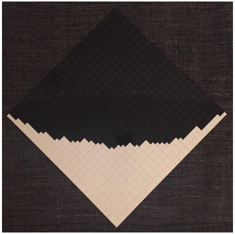
HLF: $$\dim \mathbb{S}_\lambda = f^\lambda = \frac{n!}{\prod_{\square\in\lambda} h_\square}$$



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2	2	3		
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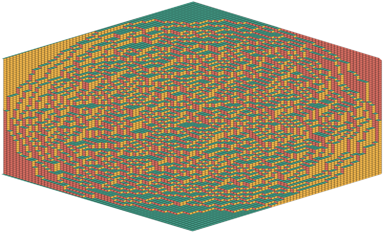
$$\dim V_\lambda = s_\lambda(1^N) = \prod_{\square\in\lambda} \frac{N + c(\square)}{h_\square}$$



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Basis for $\mathbb{S}_\lambda$ given by **SYTs** of shape λ:
$T : \lambda \xrightarrow{\sim} \{1, \ldots, n\}$ and $T_{i,j} < T_{i,j+1}, T_{i+1,j}$

$$T = \wedge \begin{array}{|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{<} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 1 & 3 & 4 & 7 & 10 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 6 & 9 \\ \cline{1-2} \end{array}$$

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$\lambda = (3, 2, 1)$:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&6\\\cline{1-2} 5\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2} 6\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&6\\\cline{1-2} 4\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&4\\\cline{1-2} 5\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5\\\cline{1-2} 4\\\cline{1-1}\end{array} + \textit{all transposed}$$

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Hook-length formula [Frame-Robinson-Thrall]:

$$f^\lambda := \#\{\text{SYTs of shape } \lambda\} = \frac{|\lambda|!}{\prod_{u \in \lambda} h_u} = \frac{6!}{5 * 3 * 3 * 1 * 1 * 1} = 16$$

Hook length of box $u = (i, j) \in \lambda$: $h_u = \lambda_i - j + \lambda'_j - i + 1 = \#■ \in$ 

Asymptotics of SYT

Standard Young Tableaux of shape λ: $\begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 \end{ytableau}$ $\begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 \end{ytableau}$ $\begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 \end{ytableau}$ $\begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \end{ytableau}$ $\begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \end{ytableau}$

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Hook-length formula [Frame-Robinson-Thrall]:

$$\#\{\text{SYTs of shape } \lambda\} = f^\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} \lambda_i - i + \lambda'_j - j + 1} = \frac{5!}{4*3*2*1*1}$$

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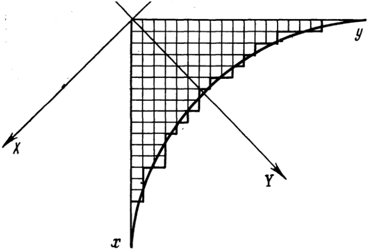

$$\sum_{\lambda\vdash n} (f^\lambda)^2 = n!$$

Theorem[Vershik-Kerov, Logan-Shepp 1977]
Under the Plancherel measure $Pr[\lambda] = \frac{(f^\lambda)^2}{n!}$, the typical partition $\lambda \vdash n$ looks as above and for them $f^\lambda = \sqrt{n!}e^{-O(\sqrt{n})}$.
Moreover, there exist c_1, c_0, such that

$$e^{-c_1\sqrt{n}}\sqrt{n!} \leq \max_{\lambda\vdash n} f^\lambda \leq e^{-c_0\sqrt{n}}\sqrt{n!}.$$

Problem: Show that there is a c, s.t. $\max_{\lambda\vdash n} f^\lambda \sim e^{-c\sqrt{n}}\sqrt{n!}$ as $n\to\infty$.

Counting skew SYTs: formulas

Outer shape λ, inner – μ, e.g. for $\lambda = (5,4,4,2,1), \mu = (2,2,1)$:

Skew SYT:

		2	3	6
		5	8	
	1	7	10	
4	9			
11				

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Jacobi-Trudi[Feit 1953]:

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)} .$$

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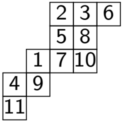

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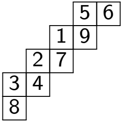
$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)}.$$

No product formula, e.g.

$\lambda/\mu = \delta_{n+2}/\delta_n$:



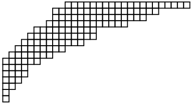
$\leftrightarrow \quad 8 > 3 < 4 > 2 < 7 > 1 < 9 > 5 < 6$

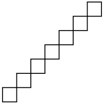
$f^{\delta_{n+2}/\delta_n} = E_{2n+1}$:

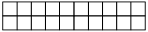
$$1 + E_1 x + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + E_4 \frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

Euler numbers: $2, 5, 16, 61 \ldots$.

Asymptotics of the number of skew SYTs

$\lambda/\mu =$  , $|\lambda/\mu| = n \to \infty$

 $\to f^{\lambda/\mu} = n!$

 $\to f^{\lambda/\emptyset} = C_{n/2} \approx 2^n \frac{2\sqrt{2}}{n^{3/2}\sqrt{\pi}}$

Question: What is the asymptotic value of $f^{\lambda/\mu}$, $|\lambda/\mu| = n$ as $n \to \infty$ and λ, μ change under various regimes:

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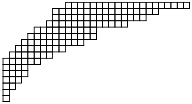
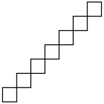
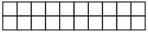
$\lambda/\mu =$  , $|\lambda/\mu| = n \to \infty$

 $\to f^{\lambda/\mu} = n!$

 $\to f^{\lambda/\emptyset} = C_{n/2} \approx 2^n \frac{2\sqrt{2}}{n^{3/2}\sqrt{\pi}}$

Question: What is the asymptotic value of $f^{\lambda/\mu}$, $|\lambda/\mu| = n$ as $n \to \infty$ and λ, μ change under various regimes:

0. If $\mu = \emptyset$, then $f^{\lambda} \sim \sqrt{n!}(1 + O(1/n))$ for $\lambda \sim$ Plancherel.

1. [Stanley, 2001, cor. to Vershik-Kerov]:
when μ is fixed, $\lambda^n \to (a; b)$ (Frobenius limit):

$$f^{\lambda^n/\mu} \sim f^{\lambda^n} s_{\mu}(\rho_a^+; \rho_b^-)(1 + O(1/n)),$$

where ρ_a^+, ρ_b^- are the corresponding specializations.
Similar results in [Corteel-Goupil-Schaeffer], [Okounkov-Olshanski].

2. [Dousse-Féray] via characters.

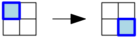
Tool: Hook-Length formula for skew shapes

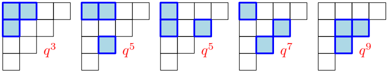
Theorem (Naruse-Ikeda)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

where $\mathcal{E}(\lambda/\mu)$ *is the set of* excited diagrams *of* λ/μ.

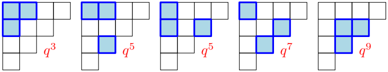
Excited diagrams:

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$



$$f^{(4321/21)} = 7! \left(\frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} \right) = 61$$

Tool: Hook-Length formula for skew shapes



$$s_{\lambda/\mu}(1,q,q^2,\ldots) = \sum_{T\in SSYT(4321/21)} q^{|T|} = \frac{q^3}{(1-q)^4(1-q^3)^3} + 2\times\frac{q^5}{(1-q)^3(1-q^3)^3(1-q^5)} + \cdots$$

Theorem (Morales-Pak-P)

For skew SSYTs, we have that

$$s_{\lambda/\mu}(1,q,q^2,\ldots) = \sum_{T\in SSYT(\lambda/\mu)} q^{|T|} = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in[\lambda]\setminus D} \left[\frac{q^{\lambda'_j-i}}{1-q^{h(i,j)}}\right].$$

Theorem (Morales-Pak-P)

For (reverse) plane partitions of skew shape λ/μ we have that

$$\sum_{\pi\in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S\in PD(\lambda/\mu)} \prod_{u\in S} \left[\frac{q^{h(u)}}{1-q^{h(u)}}\right].$$

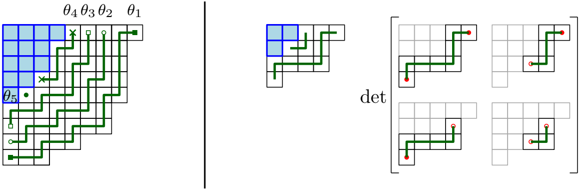
where $PD(\lambda/\mu) := \{S \subset [\lambda] : S \subset [\lambda]\setminus D,$ for some $D\in\mathcal{E}(\lambda/\mu)\}$ is the set of "pleasant diagrams".

NILP proof of NHLF

Theorem[Lascoux-Pragacz, Hamel-Goulden] If $(\theta_1,\ldots,\theta_k)$ is a Lascoux–Pragacz decomposition (i.e. maximal outer border strip decomposition) of λ/μ, then

$$s_{\lambda/\mu} = \det\left[\, s_{\theta_i\#\theta_j}\,\right]_{i,j=1}^{k}.$$

where $s_\varnothing = 1$ and $s_{\theta_i\#\theta_j} = 0$ if the $\theta_i\#\theta_j$ is undefined.
θ_1 – border strip following the inner border of λ;
θ_i – inner border of $\lambda \setminus (\theta_1 \cup \cdots \cup \theta_{i-1})$ etc until μ is hit,
then – border strips from each connected part etc.
Ordering: corners.
Strip $\theta_i\#\theta_j :=$ shape of θ_1 between the diagonals of the endpoints of θ_i and θ_j.

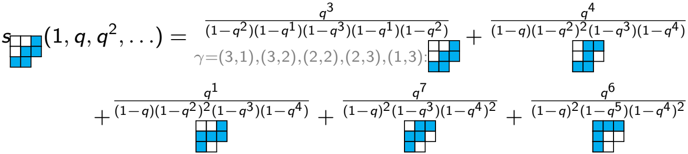


NHLF for border strips

Lemma (MPP)

For a border strip $\theta = \lambda/\mu$ with end points (a,b) and (c,d) we have

$$s_\theta(1,q,q^2,\ldots,) = \sum_{\substack{\gamma:(a,b)\to(c,d),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j-i}}{1-q^{h(i,j)}}.$$



$$s(1,q,q^2,\ldots) = \frac{q^3}{(1-q^2)(1-q^1)(1-q^3)(1-q^1)(1-q^2)} + \frac{q^4}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$$
$$+\frac{q^1}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)} + \frac{q^7}{(1-q)^2(1-q^3)(1-q^4)^2} + \frac{q^6}{(1-q)^2(1-q^5)(1-q^4)^2}$$

$\gamma=(3,1),(3,2),(2,2),(2,3),(1,3)$:

Proofs: induction on $|\lambda/\mu|$, or [multivariate] Chevalley formula for factorial Schurs.

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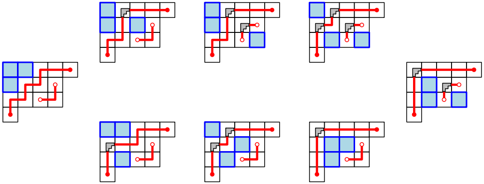
Excited diagrams for λ/μ $\leftrightarrow$ Non-Intersecting Lattice Paths:



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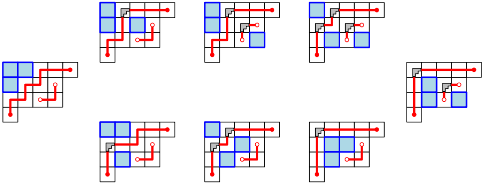
Excited diagrams for λ/μ ↔ Non-Intersecting Lattice Paths:



$$s_{\lambda/\mu} \underset{\text{Lascoux-Pragacz}}{=} \det\left[\, s_{\theta_i\#\theta_j}\,\right]_{i,j=1}^{k} \underset{s_{\text{Border Strip}}}{=} \det\left[\, \sum_{\gamma:(a_i,b_i)\to(c_j,d_j)} \prod_{u\in\gamma} \frac{q^{\cdot\cdot}}{1-q^{h_u}}\right]$$

$$\underset{Lindstrom-Gessel-Viennot}{=} \sum_{NILP:\gamma_1,\dots} \prod_{u\in\gamma_1\cup\dots} \frac{q^{\cdot\cdot}}{1-q^{h_u}} \underset{\mathcal{E}(\lambda/\mu)=NILP}{=} \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{u\in D} \frac{q^{\cdots}}{1-q^{h_u}}$$

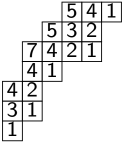
Tool: Hook-length formula

Naruse Hook-Length formula:

$$f^{\lambda/\mu} = n! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in D} \frac{1}{h_u}.$$

Define the "naive" hook-length formula:

$$F(\lambda/\mu) := n! \prod_{u \in \lambda/\mu} \frac{1}{h_u}.$$



			5	4	1
		5	3	2	
	7	4	2	1	
	4	1			
4	2				
3	1				
1					

$$F((6,5,5,3,2,2,1)/(3,2,1,1)) = \frac{17!}{5{\cdot}4{\cdot}1{\cdot}5{\cdot}3{\cdot}2{\cdot}7{\cdot}4{\cdot}2{\cdot}1{\cdot}4{\cdot}1{\cdot}4{\cdot}2{\cdot}3{\cdot}1{\cdot}1}$$

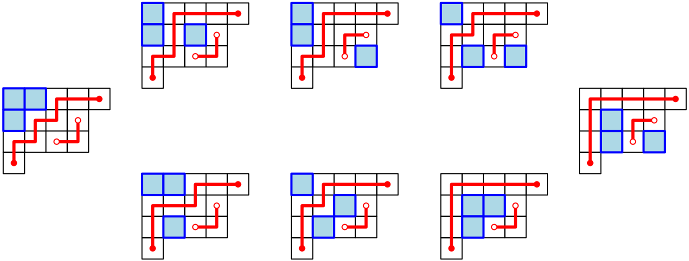
Corollary

$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\,\mathcal{E}(\lambda/\mu)|F(\lambda/\mu)$$

General bounds: size of $\mathcal{E}(\lambda/\mu)$

$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)|F(\lambda/\mu)$$

$\mathcal{E}(\lambda/\mu) = \{$ Non-intersecting Lattice Paths in λ/μ $\}$



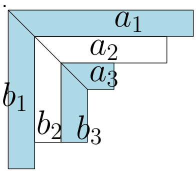
Lemma (MPP)

If $|\lambda/\mu| = n$ then $\mathcal{E}(\lambda/\mu) \leq 2^n$.

Lemma (MPP)

If d is the Durfee square size of λ, then $\mathcal{E}(\lambda/\mu) \leq n^{2d^2}$.

The “linear” regime



$a(\lambda) = (a_1, a_2, \ldots)$, $b(\lambda) = (b_1, b_2, \ldots)$ – Frobenius coordinates of λ. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta := (\beta_1, \ldots, \beta_k)$ be fixed sequences in $\mathbb{R}_+^k$.

Thoma–Vershik–Kerov (TVK) *limit* if $a_i/n \to \alpha_i$ and $b_i/n \to \beta_i$ as $n \to \infty$, for all $1 \leq i \leq k$.

Theorem (MPP)

Let $\{\lambda^{(n)}/\mu^{(n)}\}$ *be a sequence of skew shapes with a TVK limit, i.e. suppose* $\lambda^{(n)} \to (\alpha, \beta)$*, where* $\alpha_1, \beta_1 > 0$*, and* $\mu^{(n)} \to (\pi, \tau)$ *for some* $\alpha, \beta, \pi, \tau \in \mathbb{R}_+^k$*. Then*

$$\log f^{\lambda^{(n)}/\mu^{(n)}} = cn + o(n) \quad \text{as} \quad n \to \infty,$$

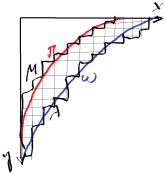
where

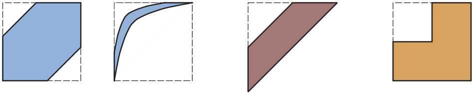
$$c = \gamma \log \gamma - \sum_{i=1}^{k} (\alpha_i - \pi_i) \log(\alpha_i - \pi_i) - \sum_{i=1}^{k} (\beta_i - \tau_i) \log(\beta_i - \tau_i)$$

and

$$\gamma = \sum_{i=1}^{k} (\alpha_i + \beta_i - \pi_i - \tau_i).$$

The stable shape: $\sqrt{n}$ scale





Theorem (MPP)

Let $\omega, \pi : [0,a] \to [0,b]$ *be continuous non-increasing functions, and suppose that area*$(\omega/\pi) = 1$*. Let* $\{\lambda^{(n)}/\mu^{(n)}\}$ *be a sequence of skew shapes with the stable shape* ω/π*, i.e.* $[\lambda^{(n)}]/\sqrt{n} \to \omega$*,* $[\mu^{(n)}]/\sqrt{n} \to \pi$*. Then*

$$\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \frac{1}{2} n \log n \quad \textit{as} \quad n \to \infty.$$

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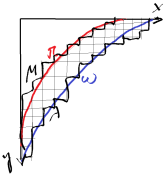

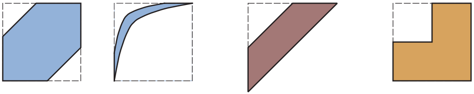

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$$\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \frac{1}{2} n \log n \quad \text{as} \quad n \to \infty.$$

Suppose $(\sqrt{N} - L)\omega \subset [\lambda^{(n)}](\sqrt{N} + L)\omega$ for some $L > 0$, similarly $\mu^{(n)}$ wrt π, then

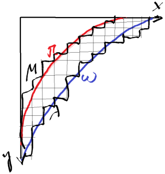
$$-\big(1+c(\omega/\pi)\big)n+o(n) \le \log f^{\lambda^{(n)}/\mu^{(n)}} - \frac{1}{2} n \log n \le -\big(1+c(\omega/\pi)\big)n + \log \mathcal{E}(\lambda^{(n)}/\mu^{(n)}) + o(n),$$

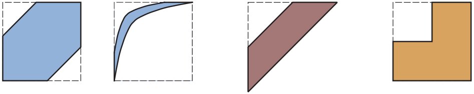
as $n \to \infty$, where

$$c(\omega/\pi) = \iint_{\omega/\pi} \log h(x, y) dxdy,$$

where $h(x, y)$ is the hook length from (x, y) to ω.

The stable shape: $\sqrt{n}$ scale





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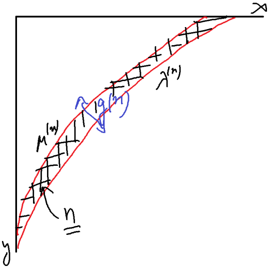
$$\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \frac{1}{2} n \log n \quad \textit{as} \quad n \to \infty.$$

In fact [Morales-Pak-Tassy, 2018],

$$\lim_{n\to\infty} \frac{\log f^{\lambda^{(n)}/\mu^{(n)}} - \frac{1}{2} n \log n}{n} \to c$$

for some constant c. Proof – variational principle for weighted lozenge tilings.
Problem: Find c for specific shapes, e.g. δ_{2k}/δ_k?

Subpolynomial depth, “thin” shapes



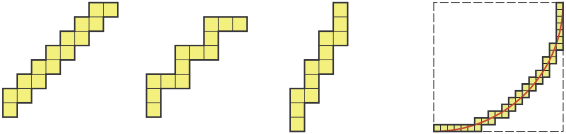
Suppose
depth$:= \max_{u\in\lambda/\mu} h_u =: g(n) = n^{o(1)}$
(subpolynomial growth).

Theorem (MPP)

Let $\{\nu_n = \lambda^{(n)}/\mu^{(n)}\}$ *be a sequence of skew partitions with a subpolynomial depth shape associated with the function* $g(n)$*. Then*

$$\log f^{\nu_n} = n\log n - \Theta\big(n\log g(n)\big) \quad \textit{as} \quad n\to\infty.$$

Thin ribbons



Zigzag: $\rho_k := \delta_{k+2}/\delta_k$, $E_n = |\{\sigma \in S_n : \sigma(1) < \sigma(2) > \sigma(3) < \cdots\}|$ – Euler numbers, alternating permutations.

$$f^{\rho_n} = E_{2n+1}; \qquad E_m \sim m!(2/\pi)^m 4/\pi(1+o(1))$$

From theorem: $F(\rho_k) = n!/3^k$, $\mathcal{E}(\rho_k) = C_k$, so

$$\frac{(2k+1)!}{3^k} \leq E_{2k+1} \leq \frac{(2k+1)!C_k}{3^k}$$

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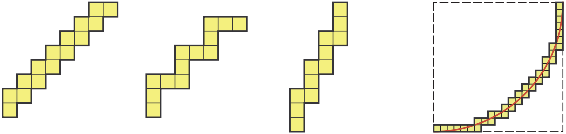

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Problem: If $\gamma_n := \lambda/\mu$ is a border strip (ribbon of thickness 1, n boxes) approaching a given curve γ under rescaling by n, what is $\log f^{\gamma_n} - n \log n$ in terms of γ? Is it true that $\frac{\log f^{\gamma_n} - n \log n}{n} \to c(\gamma)$ for some constant $c(\gamma)$? (Permutations with certain descent sequences)

Structure constants I

Tensor product of irreducible GL representations:

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}$$

Littlewood-Richardson coefficients: $c^\nu_{\lambda\mu}$

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$$s_\lambda(x)s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(x)$$

$$s_{2,1}s_{2,1} = s_{(4,2)} + s_{(4,1,1)} + s_{(3,3)} + 2s_{(3,2,1)} + s_{(3,1,1,1)} + s_{(2,2,2)} + s_{(2,2,1,1)}$$

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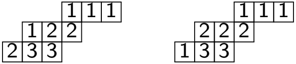
Theorem (Littlewood-Richardson, stated 1934, proven 1970's)

The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.



(LR tableaux of shape $(6,4,3)/(3,1)$ and type $(4,3,2)$. $c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)

Structure constants II

Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

$$s_\nu(x.y) = \sum_{\lambda,\mu} g(\lambda, \mu, \nu) s_\lambda(x) s_\mu(y)$$

$$s_{(2,1)}(x.y)=s_{(2,1)}(x)s_{(3)}(y)+s_{(2,1)}(x)s_{(1,1,1)}(y)+s_{(2,1)}(y)s_{(3)}(x)+s_{(2,1)}(y)s_{(1,1,1)}(x)+s_{(2,1)}(x)s_{(2,1)}(y)$$

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Plethysm coefficients: $GL_n \xrightarrow{\rho^\nu} GL_m \xrightarrow{\rho^\mu} GL_N$: $\rho^\mu \circ \rho^\nu : GL_n \to GL_N$:

$$\rho^\mu(\rho^\nu) = \bigoplus_\lambda V_\lambda^{\oplus a_\lambda(\mu[\nu])}$$

$a_\lambda(d[n])$ – multiplicity of V_λ in $Sym^d(Sym^n V)$ under GL action.

$$s_d[s_n(x)] = s_d(x_1^n, x_1^{n-1}x_2, \ldots) = \sum_\lambda a_\lambda(d[n]) s_\lambda$$

$$s_2[s_2(x)] = s_2(x_1^2, x_1x_2, x_2^2, \ldots) = s_4(x) + s_{(2,2)}(x)$$

A major problem in Algebraic Combinatorics

Problem (Murnaghan 1938, Stanley)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$*, i.e. a family of combinatorial objects* $\mathcal{O}_{\lambda,\mu,\nu}$*, s.t.* $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$. [1]

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Combinatorial formulas for $g(\lambda, \mu, \nu)$:

Two two-row partitions [*Remmel–Whitehead*, 1994; *Blasiak–Mulmuley–Sohoni*,2013] ;
One two-row and other restrctions [*Ballantine-Orellana*, 2006]
One hook $\nu = (n-k, 1^k)$ [*Blasiak 2012, Blasiak-Liu 2014, Liu 2015*]
Other special cases [Bessenrodt-Bowman, Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova, Mishna-Rosas-Sundaram, Vallejo].

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Positivity questions:

Tensor square conjecture Haide-Saxl-Tiep-Zalesski (2012): For every $n \geq 9$ there is a $\lambda \vdash n$, such that $g(\lambda, \lambda, \mu) > 0$ for all $\mu \vdash n$.

Saxl conjecture (2012): $g((k, k-1, \ldots, 1), (k, k-1, \ldots, 1), \mu) > 0$ for all $\mu \vdash \binom{k+1}{2}$.

Various partial results: Bessenrodt-Behns (2004), Pak-Panova-Vallejo (2013), Ikenmeyer (2015), Bessenrodt (2017), Luo-Sellke (2016), Li (2020), Harman-Ryba (2021), Zhao (2023+).

Geometric Complexity Theory: $g(n^d, n^d, \mu) > 0$ for certain μ's, also $g(\lambda, n^d, n^d) \geq a_\lambda(d[n])$ in the stable range. [Ikenmeyer-P'2017]

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Combinatorial interpretation of $g(\lambda, \mu, \nu) \iff$ ComputeKron $\in \#$P?

Computing $|\chi^\lambda(\pi)|$ is not in $\#$P, deciding if $\chi^\lambda(\pi) \neq 0$ is not in NP (assuming PH not collapsing) [Ikenmeyer-Pak-Panova, 2022]

Geometric complexity theory: find inequalities between certain multiplicities (close to $g(\lambda, n^d, n^d)$ and $a_\lambda(d[n])$ to show VP $\neq$ VNP. (...long list of works...)

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Asymptotics:

$g(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}) \sim ?$ as n grows for various regimes? (same for $f^{\lambda/\mu}$, $c^\lambda_{\mu\nu}$, $a_\lambda(d[n])$ etc)

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What? Why? → NEXT

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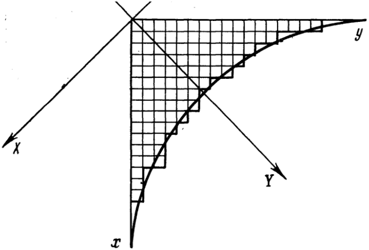
Maximal dimension



$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

Theorem (Vershik-Kerov, Logan-Shepp 1977)

Under the Plancherel measure $Pr[\lambda] = \frac{(f^\lambda)^2}{n!}$*, the typical partition* $\lambda \vdash n$ *looks as above and for them* $f^\lambda = \sqrt{n!}e^{-O(\sqrt{n})}$*.*
Moreover, there exist c_1, c_0*, such that*

$$e^{-c_1\sqrt{n}}\sqrt{n!} \leq \max_{\lambda \vdash n} f^\lambda \leq e^{-c_0\sqrt{n}}\sqrt{n!}.$$

Maximal multiplicities

Theorem [Stanley]

$$\max_{\lambda \vdash n} \max_{\mu \vdash n} \max_{\nu \vdash n} \; g(\lambda, \mu, \nu) \;=\; \sqrt{n!}\, e^{-O(\sqrt{n})},$$

$$\max_{0 \le k \le n} \max_{\lambda \vdash n} \max_{\mu \vdash k} \max_{\nu \vdash n-k} \; c^{\lambda}_{\mu,\nu} \;=\; 2^{n/2 - O(\sqrt{n})}.$$

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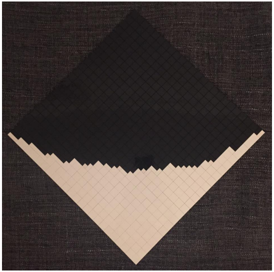
Theorem (Pak-Panova-Yeliussizov'18)

Let $\{\lambda^{(n)}\vdash n\}$, $\{\mu^{(n)}\vdash n\}$, $\{\nu^{(n)}\vdash n\}$ *be three partition sequences, such that*

$$(*) \qquad g\big(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}\big) \;=\; \sqrt{n!}\, e^{-O(\sqrt{n})}\,.$$

Then $\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}$ *are Vershik-Kerov-Logan-Shepp shape. Conversely, for every two VKLS-shape sequences* $\{\lambda^{(n)}\vdash n\}$ *and* $\{\mu^{(n)}\vdash n\}$*, there exists a VKLS sequence* $\{\nu^{(n)}\vdash n\}$*, s.t. (*) holds.*

VKLS shape:



$\mathbf{D}(n) := \max_{\lambda\vdash n} f^{\lambda}$

Largest Kroneckers: proofs

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Identities:

$$\sum_{\lambda,\mu,\nu\vdash n} g(\lambda,\mu,\nu)^2 = \sum_{\alpha\vdash n} z_\alpha \geq z_{1^n} = n!,$$

where $z_\alpha = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$ when $\alpha = (1^{m_1} 2^{m_2} \ldots)$,
so $\max g(\lambda,\mu,\nu) \geq \sqrt{n!/p(n)^3} = e^{\frac{1}{2} n \log(n) - \frac{1}{2} n - O(\sqrt{n})}$.

Largest Kroneckers: proofs

Theorem (Pak-Panova-Yeliussizov'18)

Let $\{\lambda^{(n)} \vdash n\}$, $\{\mu^{(n)} \vdash n\}$, $\{\nu^{(n)} \vdash n\}$ *be three partition sequences, such that*

$$(*) \qquad g\big(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}\big) \, = \, \sqrt{n!}\, e^{-O(\sqrt{n})}\,.$$

Then $\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}$ *are Vershik-Kerov-Logan-Shepp shape. Conversely, for every two VKLS-shape sequences* $\{\lambda^{(n)} \vdash n\}$ *and* $\{\mu^{(n)} \vdash n\}$*, there exists a VKLS sequence* $\{\nu^{(n)} \vdash n\}$*, s.t. (*) holds.*

Identities:

$$\sum_{\lambda,\mu,\nu\vdash n} g(\lambda,\mu,\nu)^2 = \sum_{\alpha\vdash n} z_\alpha \geq z_{1^n} = n!,$$

where $z_\alpha = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$ when $\alpha = (1^{m_1} 2^{m_2} \ldots)$,
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Conversely, since

$$\sum_\nu g(\lambda,\mu,\nu) f^\nu = f^\lambda f^\mu$$

at least one of the summands on the LHS must be $\sim n! e^{-O(\sqrt{n})}$, this is $\nu^{(n)}$.

Littlewood-Richardson

Theorem (PPY'18)

There exists a constant $d > 0$, s.t. for all $n > k \geq 1$:

$$\sqrt{\binom{n}{k}}\, e^{-d\sqrt{n}} \leq \max_{\lambda \vdash n} \max_{\mu \vdash k} \max_{\nu \vdash n-k} c^{\lambda}_{\mu,\nu} \leq \sqrt{\binom{n}{k}}.$$

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Fix $0 < \theta < 1$ and let $k_n := \lfloor \theta n \rfloor$. Then:

1. for every VKLS sequence $\{\lambda^{(n)} \vdash n\}$, there exist VKLS sequences $\{\mu^{(n)} \vdash k_n\}$ and $\{\nu^{(n)} \vdash n - k_n\}$, s.t.

$$c^{\lambda^{(n)}}_{\mu^{(n)},\, \nu^{(n)}} = \binom{n}{k_n}^{1/2} e^{-O(\sqrt{n})},$$

2. For all VKLS sequences $\{\mu^{(n)} \vdash k_n\}$ and $\{\nu^{(n)} \vdash n - k_n\}$, there exists a VKLS sequence $\{\lambda^{(n)} \vdash n\}$, s.t. $(**)$ holds,
3. For all VKLS sequences $\{\lambda^{(n)} \vdash n\}$ and $\{\mu^{(n)} \vdash k_n\}$, there exists a sequence $\{\nu^{(n)} \vdash n - k_n\}$, s.t.

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[Belinschi-Guionnet-Huang'20+]: General upper bounds on $c^{\lambda}_{\mu\nu}$ for "nice measures" via elliptical [random matrix] integrals.

Large Littlewood-Richardson: identities, inequalities

$$\sum_{\lambda\vdash n} c^{\lambda}_{\mu,\nu}\, f^{\lambda} \;=\; \binom{n}{k} f^{\mu} f^{\nu} \qquad \text{and} \qquad \sum_{\mu\vdash k,\,\nu\vdash n-k} c^{\lambda}_{\mu,\nu}\, f^{\mu} f^{\nu} \;=\; f^{\lambda}.$$

Lemma: For every $0 \leq k \leq n$, we have:

$$\sum_{\lambda\vdash n} \sum_{\mu\vdash k,\,\nu\vdash n-k} \left(c^{\lambda}_{\mu,\nu}\right)^2 \;\geq\; \binom{n}{k}.$$

which follows from $c^{\lambda}_{\mu\nu} = \langle \chi^{\lambda} \downarrow^{S_n}_{S_k\times S_{n-k}}, \chi^{\mu}\chi^{\nu}\rangle$ giving

$$\sum_{\lambda\vdash n} \sum_{\mu\vdash k,\,\nu\vdash n-k} \left(c^{\lambda}_{\mu,\nu}\right)^2 = \cdots = \sum_{\alpha\in Conj(S_k\times S_{n-k})} \frac{z_{\alpha}(S_n)}{z_{\alpha}(S_k \times S_{n-k})} \geq \frac{z_{1^n}}{z_{1^k} z_{1^{n-k}}}$$

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Other inequalities:

$$\sum_{\lambda\vdash n} \left(c^{\lambda}_{\mu,\nu}\right)^2 \le \sum_{\lambda\vdash n} c^{\lambda}_{\mu,\nu}\, \frac{f^{\lambda}}{f^{\mu} f^{\nu}} \;=\; \frac{1}{f^{\mu} f^{\nu}} \cdot f^{\mu} f^{\nu} \binom{n}{k} \;=\; \binom{n}{k},$$

$$\sum_{\mu\vdash k,\,\nu\vdash n-k} \left(c^{\lambda}_{\mu,\nu}\right)^2 \le \sum_{\mu\vdash k,\,\nu\vdash n-k} c^{\lambda}_{\mu,\nu}\, \frac{f^{\mu} f^{\nu}\binom{n}{k}}{f^{\lambda}} \;=\; \frac{1}{f^{\lambda}} \cdot f^{\lambda}\binom{n}{k} \;=\; \binom{n}{k},$$

Small number of rows

Theorem (Pak-P'20)

Let $\lambda, \mu, \nu \vdash n$ such that $\ell(\lambda) = \ell$, $\ell(\mu) = m$, and $\ell(\nu) = r$. Then:

$$g(\lambda, \mu, \nu) \le \left(1 + \frac{\ell m r}{n}\right)^{n} \left(1 + \frac{n}{\ell m r}\right)^{\ell m r}.$$

Corollary: Let $\lambda = (\ell^2)^{\ell}$, where $\ell = \sqrt[3]{n}$, then

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Proof via contingency arrays:

$$T(\lambda, \mu, \nu) = \#\{(X_{i,j,k}) \in \mathbb{Z}_{\geq 0}^{\ell m r} : \sum_{j=1,k=1}^{m,r} X_{i,j,k} = \lambda_i, \sum_{i=1,k=1}^{\ell,r} X_{i,j,k} = \mu_j, \sum_{i=1,j=1}^{\ell,m} X_{i,j,k} = \nu_k\},$$

$$\sum_{\lambda,\mu,\nu} g(\lambda, \mu, \nu)\, s_\lambda(x)\, s_\mu(y)\, s_\nu(z) = \sum_{\alpha,\beta,\gamma} T(\alpha, \beta, \gamma)\, x^\alpha\, y^\beta\, z^\gamma.$$

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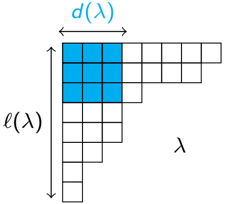
[Barvinok]: The number of 3d contingency tables with marginals (α, β, γ) is

$$\le \exp\left(\max_{Z \in P(\alpha,\beta,\gamma)} \sum_{i,j,k} (Z_{ijk} + 1) \log(Z_{ijk} + 1) - Z_{ijk} \log(Z_{ijk}) \right)$$

Small Durfee squares



Theorem (Pak-P'22)

Let $n, k \geq 1$, *and let* $\lambda, \mu, \nu \vdash n$, *such that* $d(\lambda), d(\mu), d(\nu) \leq k$. *Then:*

$$g(\lambda,\mu,\nu) \leq \frac{1}{k^{8k^2}\, 2^{8k^3}}\ n^{4k^3+13k^2+31k} = n^{4k^3+O(k^2)}.$$

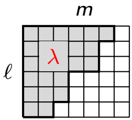
Theorem (Pak-P'22)

For all $k \geq 1$, *there is a constant* $C_k > 0$, *such that for all* $n \geq 1$

$$\max_{\substack{\lambda,\mu,\nu\vdash n \\ \ell(\lambda),\ell(\mu),\ell(\nu)\leq k}} \{ g(\lambda,\mu,\nu)\} \geq C_k\, n^{k^3-3k^2-3k+3} = n^{k^3-O(k^2)}$$

Tight asymptotics of Kronecker coefficients

$p_n(\ell, m) := \#\{\lambda \vdash n;\ \lambda \subset (m^\ell)\}$



$$\sum_{k \geq 0} p_n(\ell, m) q^n \ = \ \begin{bmatrix} m+\ell \\ m \end{bmatrix}_q$$

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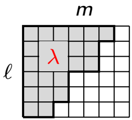

Theorem (Pak-P'15)

For all $m \geq \ell \geq 8$ *and* $2 \leq k \leq \ell m/2$, *let* $s = \min\{2k, \ell^2\}$. *We have:*

$$g(m^\ell, m^\ell, (m\ell - k, k)) = p_k(\ell, m) \ - \ p_{k-1}(\ell, m) \ > \ 0.004 \, \frac{2^{\sqrt{s}}}{s^{9/4}} \, .$$

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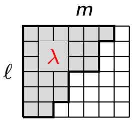

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Theorem (Melczer-P-Pemantle'19)

Let $A := \frac{\ell}{m}$ $B := \frac{n-1}{m^2}$. *Let* c, d *be solutions of [a system of integral equations]*

$$p_n(\ell, m) - p_{n-1}(\ell, m) \sim \frac{d}{m} p_{n-1}(\ell, m) \sim \frac{d\ e^{m\left[cA + 2dB - \log(1-e^{-c-d})\right]}}{2\pi m^3 \sqrt{D}}.$$

Proof via tilted geometric Random Variables.

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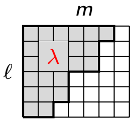

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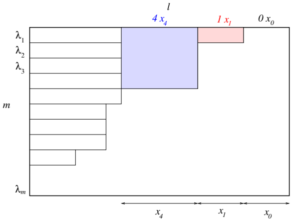
Local Central Limit Theorem



Independent geometric random variables

$x_i = \lambda_{i+1} - \lambda_i$ – part sizes

$X_i \sim Geom(p_i)$:

$$\Pr(X_i = k) = p_i q_i^k$$

$$1 - p_i = q_i = e^{-c_m - d_m i}$$

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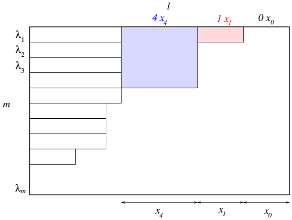

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$$n \quad = \mathbb{E}[T_m := \textstyle\sum_i iX_i] = \quad m \sum_{j=0}^{m} \frac{j/m}{1 - e^{-c_m - d_m j/m}} - \frac{m(m+1)}{2}$$

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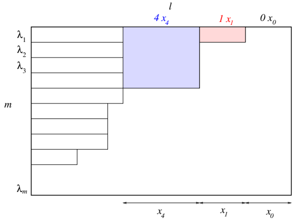

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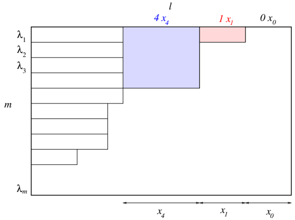
$$\Longrightarrow \Pr[X = x] = \prod p_i \exp(-c_m \ell - d_m n)$$

$$\Longrightarrow \Pr[S_m = \ell,\, T_m = n] = p_n(m,\ell) \Pr[X = x] = p_n(m,\ell) e^{-c_m A m - d_m m B} \prod p_i$$

Local Central Limit Theorem



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LCLT: (S_m, T_m) " $\to$ " Gaussian and in particular

$$\Pr(S_m = \mathbb{E}[S_m], T_m = \mathbb{E}[T_m]) \sim \frac{1}{2\pi \sqrt{\det Cov(S_m, T_m)}}.$$

Tight asymptotic results

Given $A \geq 2B > 0$, define c, d as the unique solutions to the equations:

$$A = \int_0^1 \frac{1}{1-e^{-c-td}}\,dt - 1 \qquad B = \int_0^1 \frac{t}{1-e^{-c-td}}\,dt - \frac{1}{2}\,, \tag{1}$$

Lemma

For any $A > 0$ and $B \in (0, A/2)$ there exist unique $c, d > 0$ satisfying Equations (1). Moreover, for a fixed A, when B decreases from $A/2$ to 0 then d increases strictly from 0 to ∞ and c decreases strictly from $\log\left(\frac{A+1}{A}\right)$ to 1. When $B > 0$ is fixed and A goes to ∞ then c goes to 0 and d goes to the root of $d^2 = B\left(d\log(1-e^{-d}) - dilog(1-e^{-d})\right)$.

Theorem (Melczer–Panova–Pemantle, 2018)

Given m, ℓ and n, let $A := \ell/m$ and $B := n/m^2$ and define c, d and Δ as above. Let K be any compact subset of $\{(x,y) : x \geq 2y > 0\}$. As $m \to \infty$ with ℓ and n varying so that (A, B) remains in K,

$$N_n(\ell, m) \sim \frac{e^{m\left[cA+2dB-\log(1-e^{-c-d})\right]}}{2\pi m^2\sqrt{\Delta\,(1-e^{-c})\,(1-e^{-c-d})}}\,,$$

where c and d vary in a Lipschitz manner with $(A, B) \in K$,

$$\Delta := \frac{2Be^c(e^d-1)+2A(e^c-1)-1}{d^2(e^{d+c}-1)(e^c-1)} - \frac{A^2}{d^2}$$

Consecutive differences

Theorem (Melczer–Panova–Pemantle, 2018)

Given m, ℓ *and* n, *let* $A := \ell/m$ *and* $B := n/m^2$ *and define* d *as above. Suppose* $m, \ell, n \to \infty$, *so that* (A, B) *remains in a compact subset of* $\{(x, y) : x \geq 2y > 0\}$ *and* $m^{-1}|n - \ell m/2| \to \infty$. *Then for the consecutive difference of* p_n *and for the Kronecker coefficient we have*

$$g((m\ell - n - 1, n + 1), m^{\ell}, m^{\ell}) = p_{n+1}(\ell, m) - p_n(\ell, m) \sim \frac{d}{m} p_n(\ell, m).$$

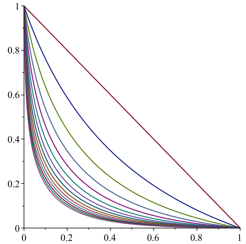
The condition $m^{-1}|n - \ell m/2| \to \infty$ is equivalent to $m|A - B/2| \to \infty$ and also to $d \notin O(m^{-1})$. It is automatically satisfied whenever (A, B) is in a compact subset of $\{(x, y) : x > 2y > 0\}$.

Limit shape: $1 = (1-e^{-c})e^{d(A-y)} + e^{-c}e^{-dx}$

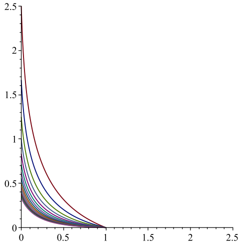
$$\lambda_i = \ell - (X_0 + X_1 + \cdots + X_{i-1}) \Longrightarrow \mathbb{E}[\lambda_i] = \ell - \sum_{j=0}^{i-1}(1/p_j - 1)$$

Set $x = i/m$, approximate the sum by an integral as $m \to \infty$ and get the equation:

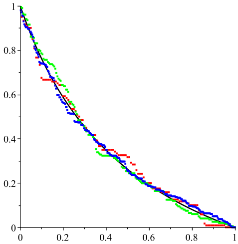
$$y := \mathbb{E}\left[\frac{\lambda_i}{m}\right] = A + x - \int_0^x \frac{1}{1-e^{-c-td}}dt = A + x - \frac{1}{d}\ln\left(\frac{e^{xd+c}-1}{e^c-1}\right).$$



$(A, B) = (1, 1/k)$ for $k = 2, \ldots, 15$



$(A, B) = (5/k, 1/k)$ for $k = 2, \ldots, 15$



Sample

The elusive lower bound

[Pak-Panova, 2014] : $g(\lambda, \lambda, \mu) \geq |\chi^{\mu}(2\lambda_1 - 1, 2\lambda_2 - 3, \ldots)|$ for $\lambda = \lambda'$

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Theorem (Pak-P'22)

For all $\varepsilon > 0$, we have:

$$\log \max \left\{ g(\lambda,\lambda,\lambda) \ : \ \lambda \vdash n, \ \lambda = \lambda' \right\} \geq \frac{1}{(16 + \varepsilon)} n \log n - O(n).$$

Thank you!