Free boundary Schur process

Mirjana Vuletić
University of Massachusetts Boston

based on joint work with
D. Betea, J. Bouttier and P. Nejjar

Dimers2023
July 2023

Outline

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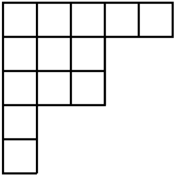
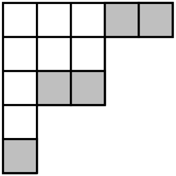
Partitions

- $\lambda = (5, 3, 3, 1, 1)$



- skew diagram λ/μ – the difference in Young diagrams of λ and μ, for $\lambda \supset \mu$
- $\lambda \succ \mu$ – interlacing condition: $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \cdots$
- $\lambda \succ \mu$ means λ/μ is a horizontal strip



- $|\lambda/\mu|$ – size of λ/μ (area)
- $s_{\lambda/\mu}$ - symmetric skew Schur function

Skew Schur functions $s_{\lambda/\mu}$

- single variable: $s_{\lambda/\mu}(x) = \begin{cases} x^{|\lambda/\mu|} & \text{if } \lambda \succ \mu, \\ 0 & \text{otherwise.} \end{cases}$
- Cauchy identity

$$\sum_{\nu} s_{\nu/\lambda}(x) s_{\nu/\mu}(y) = \frac{1}{1-xy} \sum_{\kappa} s_{\lambda/\kappa}(y) s_{\mu/\kappa}(x)$$

$$= \sum_{G=0}^{\infty} (xy)^G \sum_{\kappa} s_{\lambda/\kappa}(y) s_{\mu/\kappa}(x)$$

- Fomin growth diagram:
- λ, μ are fixed, $(\kappa, G) \to \nu$ is a bijection

$$\begin{array}{ccc} \lambda & & \\ \curlyvee & G & \\ \kappa & \prec & \mu \end{array} \longrightarrow \begin{array}{ccc} \lambda & \prec & \nu \\ & & \curlyvee \\ & & \mu \end{array} \qquad \begin{array}{ccc} \lambda & \longrightarrow & \nu \\ \downarrow & G & \downarrow \\ \kappa & \longrightarrow & \mu \end{array}$$

- $|\lambda| + |\mu| = |\kappa| + |\nu| + G$

- *nonnegative* specializations
  [Aissen, Edrei, Schoenberg, Whitney; also Thoma]

$$H(\rho; z) = e^{\gamma z} \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z} \text{ where } H(\rho; z) := \sum_{n \geq 0} h_n(\rho) z^n.$$

  where $\gamma, \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots \geq 0$ and $\sum \alpha_i + \beta_i < \infty$
- single-variable specialization

$$s_{\lambda/\mu}(\alpha_1) = \begin{cases} \alpha_1^{|\lambda/\mu|} & \text{if } \lambda \succ \mu, \\ 0 & \text{otherwise.} \end{cases}$$

- exponential specialization

$$s_{\lambda/\mu}(\exp_\gamma) = \frac{\gamma^{|\lambda/\mu|} f^{\lambda/\mu}}{|\lambda/\mu|!}$$

$$s_{\lambda/\mu}(\exp_\gamma) = \lim_{n \to \infty} s_{\lambda/\mu}\Big(\underbrace{\frac{\gamma}{n}, \ldots, \frac{\gamma}{n}}_{n \text{ variables}}\Big)$$

Outline

Free boundary Schur process

- The *free boundary Schur process* of length N is a measure over the set of sequences of partitions of the form

$$\mu^{(0)} \prec \lambda^{(1)} \succ \mu^{(1)} \prec \cdots \succ \mu^{(N-1)} \prec \lambda^{(N)} \succ \mu^{(N)}$$

which assigns to any such sequence $(\vec{\lambda}, \vec{\mu})$

$$\mathcal{W}(\vec{\lambda}, \vec{\mu}) := u^{|\mu^{(0)}|} v^{|\mu^{(N)}|} \prod_{k=1}^{N} \left(s_{\lambda^{(k)}/\mu^{(k-1)}} \left(\rho_k^+\right) s_{\lambda^{(k)}/\mu^{(k)}} \left(\rho_k^-\right) \right).$$

- $s_{\lambda/\mu}$-skew Schur functions, parameters: u, v-complex numbers and $\rho_k^{\pm}$-specializations
- the pattern $\prec, \succ, \prec, \succ, \ldots$ can be replaced with any sequence of $\prec$ and $\succ$ if we use trivial specializations $s_{\lambda/\mu}(0, \ldots) = \delta_{\lambda=\mu}$
- $q^{vol} = q^{\sum_i |\lambda(i)|}$ for the single-var. spec. $s_{\lambda/\mu}(x_1) = x_1^{|\lambda|-|\mu|}$

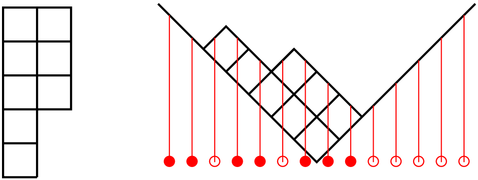
Original Schur process and its variants

$$\mathcal{W}(\vec{\lambda}, \vec{\mu}) = u^{|\mu^{(0)}|} v^{|\mu^{(N)}|} \prod_{k=1}^{N} \left(s_{\lambda^{(k)}/\mu^{(k-1)}} \left(\rho_k^+\right) s_{\lambda^{(k)}/\mu^{(k)}} \left(\rho_k^-\right) \right)$$

- $u = v = 0$: the original Schur process [Okounkov-Reshetikhin]
- $u = 1$ and $v = 0$: symmetric Schur process, also called Pfaffian Schur process [Borodin-Rains]
- $\mu^{(0)} = \mu^{(N)}$: periodic Schur process [Borodin] (factor $u^{|\mu^{(0)}|}$ for $|u| < 1$ makes the measure finite)
- no restriction on $\mu^{(0)}$ and $\mu^{(N)}$: free boundary Schur process (factor $u^{|\mu^{(0)}|} v^{|\mu^{(N)}|}$ for $|u| < 1, |v| < 1$ makes the measure finite)

Schur process as a point process

- Maya diagram



- Schur procsess is a point process on $[1, \ldots, \mathbb{N}] \times \mathbb{Z}'$, where $\mathbb{Z}' = \mathbb{Z} + 1/2$ and $\vec{\lambda}$ is represented by a point configuration

$$\mathfrak{S}(\vec{\lambda}) := \left\{ \left(i, \lambda_j^{(i)} - j + \frac{1}{2} \right), 1 \leq i \leq N, j \geq 1 \right\}$$

- For $U = \{(i_1, k_1), \ldots, (i_n, k_n)\} \subset [1, \ldots, N] \times \mathbb{Z}'$ the correlation function $\rho(U)$ is the sum of $Prob(\vec{\lambda})$ over all sequences $\vec{\lambda}$ such that k_j belongs to the Maya diagram of $\lambda^{(i_j)}, \forall j = 1, \ldots, n$.

Outline

Fock space [Miwa, Jimbo, Date]

- orthonormal basis $|\lambda\rangle, \lambda \in \mathcal{P}$-set of all partitions
- *vertex operators* $\Gamma_{\pm}(\rho)$ by

$$\langle\lambda|\Gamma_{+}(\rho)|\mu\rangle = \langle\mu|\Gamma_{-}(\rho)|\lambda\rangle = s_{\mu/\lambda}(\rho), \qquad \lambda, \mu \in \mathcal{P}.$$

- *fermionic operators* ψ_k and ψ_k^* for $k \in \mathbb{Z}'$ satisfying

$$\psi_k \psi_k^* |\lambda\rangle = \begin{cases} |\lambda\rangle & \text{if } k \in \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

- *free boundary states*

$$|\underline{v}\rangle := \sum_{\lambda \in \mathcal{P}} v^{|\lambda|} |\lambda\rangle, \qquad \langle\underline{u}| := \sum_{\lambda \in \mathcal{P}} u^{|\lambda|} \langle\lambda|, \qquad u, v \in \mathbb{C}.$$

Partition function and correlation function

Partition function can be written as

$$Z = \langle \underline{u} | \Gamma_+(\rho_1^+)\Gamma_-(\rho_1^-)\cdots\Gamma_+(\rho_N^+)\Gamma_-(\rho_N^-)|\underline{v}\rangle.$$

Correlation function $\rho(U)$ for $U = \{(i_1, k_1), \ldots, (i_n, k_n)\}$ can be written as

$$\frac{1}{Z}\langle \underline{u}| \cdots \Gamma_+(\rho_{i_1}^+)\psi_{k_1}\psi_{k_1}^*\Gamma_-(\rho_{i_1}^-)\cdots\Gamma_+(\rho_{i_n}^+)\psi_{k_n}\psi_{k_n}^*\Gamma_-(\rho_{i_n}^-)\cdots|\underline{v}\rangle.$$

Partition function

- $\Gamma_+(\rho)|\emptyset\rangle = |\emptyset\rangle \qquad \langle\emptyset|\Gamma_-(\rho) = \langle\emptyset|$
- case $u = v = 0$

$$\langle\emptyset|\Gamma_+(\rho_1^+)\Gamma_-(\rho_1^-)\cdots\Gamma_+(\rho_N^+)\Gamma_-(\rho_N^-)|\emptyset\rangle = \prod_{1\le k\le \ell\le N} H(\rho_k^+;\rho_\ell^-)$$

- we use commutation relations for vertex operators (Cauchy identity):

$$\Gamma_+(\rho)\Gamma_-(\rho') = H(\rho;\rho')\Gamma_-(\rho')\Gamma_+(\rho)$$

$$H(\rho;\rho') := \sum_{\lambda\in\mathcal{P}} s_\lambda(\rho)s_\lambda(\rho') = \prod_{i,j}\frac{1}{1-x_i x_j}$$

Partition function

- case $v = 0$: $\langle \underline{u} | \Gamma_+(\rho_1^+)\Gamma_-(\rho_1^-) \cdots \Gamma_+(\rho_N^+)\Gamma_-(\rho_N^-) | \emptyset \rangle$

$$\Big(\prod_{1 \le k \le \ell \le N} H(\rho_k^+; \rho_\ell^-) \Big) \tilde{H}(u\rho^-)$$

- reflection relations for vertex operators (Littlewood identity)

$$\Gamma_+(\rho)|\underline{v}\rangle = \tilde{H}(v\rho)\Gamma_-(v^2\rho)|\underline{v}\rangle \qquad \langle \underline{u}|\Gamma_-(\rho) = \tilde{H}(u\rho)\langle \underline{u}|\Gamma_+(u^2\rho)$$

$$\tilde{H}(\rho) := \sum_{\lambda \in \mathcal{P}} s_\lambda(\rho) = \prod_i \frac{1}{1 - x_i}$$

- general case: $\langle \underline{u} | \Gamma_+(\rho_1^+)\Gamma_-(\rho_1^-) \cdots \Gamma_+(\rho_N^+)\Gamma_-(\rho_N^-) | \underline{v} \rangle$

$$\prod_{1 \le k \le \ell \le N} H(\rho_k^+; \rho_\ell^-) \prod_{n \ge 1} \frac{\tilde{H}(u^{n-1}v^n\rho^+)\tilde{H}(u^n v^{n-1}\rho^-)H(u^{2n}\rho^+; v^{2n}\rho^-)}{1 - u^n v^n}$$

Correlation function

Correlation function can be written as

$$\rho(U) = \frac{1}{Z} \langle \underline{u} | \cdots \Gamma_+(\rho^+_{i_1}) \psi_{k_1} \psi^*_{k_1} \Gamma_-(\rho^-_{i_1}) \cdots \Gamma_+(\rho^+_{i_n}) \psi_{k_n} \psi^*_{k_n} \Gamma_-(\rho^-_{i_n}) \cdots | \underline{v} \rangle,$$

- Commutation relations:

$$\Gamma_\pm(\rho)\psi(z) = H(\rho; z^{\pm 1})\psi(z)\Gamma_\pm(\rho),$$

$$\Gamma_\pm(\rho)\psi^*(w) = H(\rho; w^{\pm 1})^{-1}\psi^*(w)\Gamma_\pm(\rho).$$

- $\langle \underline{u} | \psi_1 \psi^*_1 \cdots \psi_n \psi^*_n | \underline{v} \rangle = ?$
- case $u = v = 0$

$$\langle \emptyset | \psi_1 \psi^*_1 \cdots \psi_n \psi^*_n | \emptyset \rangle = \det A$$

where

$$A_{ij} = \begin{cases} \langle \emptyset | \psi_i \psi^*_j | \emptyset \rangle & i \geq j \\ \langle \emptyset | \psi^*_j \psi^*_i | \emptyset \rangle & i < j \end{cases}$$

Wick's formula

- case $v = 0$

Wick's formula

Let Ψ be the vector space spanned by the ψ_k and ψ_k^*, $k \in \mathbb{Z}'$. For $\phi_1, \ldots, \phi_{2n} \in \Psi$, we have

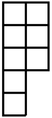
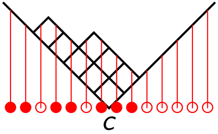
$$\langle \underline{u} | \phi_1 \cdots \phi_{2n} | \emptyset \rangle = \text{pf } A$$

where A is the antisymmetric matrix defined by $A_{ij} = \langle \underline{u} | \phi_i \phi_j | \emptyset \rangle$ for $i < j$.

- free boundary case $\langle \underline{u} | \phi_1 \cdots \phi_{2n} | \underline{v} \rangle$- we need more general free boundary states

Shift-mixed process - extended free boundary states

- shift-mixed periodic Schur process [Borodin]

- charged Maya diagram  

- extended free boundary states

$$|\underline{v,t}\rangle = \sum_{(\lambda,c)\in\mathcal{P}\times 2\mathbb{Z}} t^{c/2} v^{|\lambda|+c^2/2} |\lambda, c\rangle,$$

$$\langle \underline{u,t}| = \sum_{(\lambda,c)\in\mathcal{P}\times 2\mathbb{Z}} t^{c/2} u^{|\lambda|+c^2/2} \langle\lambda, c|.$$

- $|\underline{v,t}\rangle := e^{X(v,t)}|\emptyset\rangle, \qquad X(v,t) = \sum_{\substack{(k,\ell)\in\mathbb{Z}'^2 \\ k>\ell}} \tilde{\psi}_k(v,t)\tilde{\psi}_\ell(v,t)$

$$\tilde{\psi}_i(v,t) = \begin{cases} t^{1/2} v^i \psi_i & \text{for } i \in \mathbb{Z}'_{>0}, \\ (-1)^{i+1/2} t^{-1/2} v^{-i} \psi_i^* & \text{for } i \in \mathbb{Z}'_{<0}. \end{cases}$$

The shift-mixed correlation function

The shift-mixed correlation function $\hat{\rho}(U)$ can be written as

$$\frac{1}{\widehat{Z}}\langle \underline{u,t}| \cdots \Gamma_+(\rho_{i_1}^+)\psi_{k_1}\psi_{k_1}^*\Gamma_-(\rho_{i_1}^-)\cdots\Gamma_+(\rho_{i_n}^+)\psi_{k_n}\psi_{k_n}^*\Gamma_-(\rho_{i_n}^-)\cdots|\underline{v,t}\rangle.$$

Wick's formula for extended free boundary states

Let Ψ be the vector space spanned by the ψ_k and ψ_k^*, $k \in \mathbb{Z}'$. For $\phi_1, \ldots, \phi_{2n} \in \Psi$, we have

$$\frac{\langle \underline{u,t}|\phi_1 \cdots \phi_{2n}|\underline{v,t}\rangle}{\langle \underline{u,t}|\underline{v,t}\rangle} = \operatorname{pf} A$$

where A is the antisymmetric matrix defined by $A_{ij} = \langle \underline{u,t}|\phi_i\phi_j|\underline{v,t}\rangle / \langle \underline{u,t}|\underline{v,t}\rangle$ for $i < j$.

The shift-mixed correlation function

Pfaffian correlations:

$$\hat{\rho}(U) = \mathrm{Pf}\, K$$

where $K(i, k; i', k')$ is equal to

$$\begin{bmatrix} [z^k w^{k'}] F(i,z) F(i',w) \langle \underline{u,t} | \psi(z)\psi(w) | \underline{v,t} \rangle & \left[\frac{z^k}{w^{k'}}\right] \frac{F(i,z)}{F(i',w)} \langle \underline{u,t} | \psi(z)\psi^*(w) | \underline{v,t} \rangle \\ \left[\frac{w^{k'}}{z^k}\right] \frac{F(i',w)}{F(i,z)} \langle \underline{u,t} | \psi^*(z)\psi(w) | \underline{v,t} \rangle & \left[\frac{1}{z^k w^{k'}}\right] \frac{1}{F(i,z)F(i',w)} \langle \underline{u,t} | \psi^*(z)\psi^*(w) | \underline{v,t} \rangle \end{bmatrix},$$

where

- $F(i, z)$ is coming from the commutation relations; depends on u, v and ρ's
- fermionic propagator - the four expectations of the form $\langle \underline{u, t} | \psi(z)\psi(w) | \underline{v, t} \rangle$ can be written explicitly in terms of u, v and t

Explicit formulas

$$F(i,z)=\frac{\prod_{1\le \ell\le i} H(\rho_\ell^+;z)}{\prod_{i\le \ell\le N} H(\rho_\ell^-;z^{-1})}\cdot\prod_{n\ge1}\frac{H(u^{2n}v^{2n-2}\rho^-;z)H(u^{2n}v^{2n}\rho^+;z)}{H(u^{2n-2}v^{2n}\rho^+;z^{-1})H(u^{2n}v^{2n}\rho^-;z^{-1})}$$

$$\kappa_{1,1}(z,w)=\frac{v^2}{tz^{1/2}w^{3/2}}\cdot\frac{((uv)^2;(uv)^2)_\infty^2}{(uz,uw,-\frac{v}{z},-\frac{v}{w};uv)_\infty}\cdot\frac{\theta_{(uv)^2}(\frac{w}{z})}{\theta_{(uv)^2}(u^2zw)}\cdot\frac{\theta_3\left((\frac{tzw}{v^2})^2;(uv)^4\right)}{\theta_3(t^2;(uv)^4)}$$

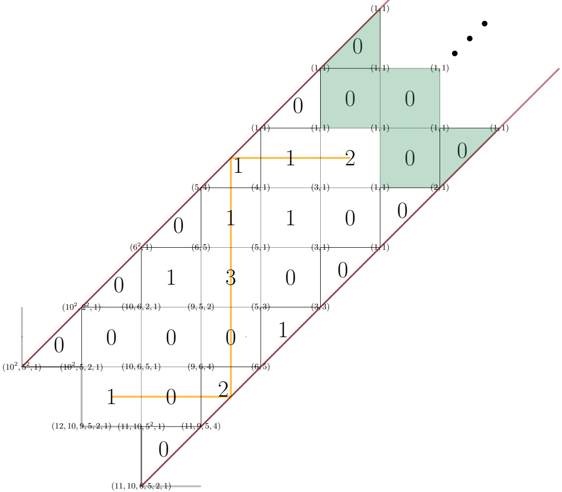
$$\kappa_{1,2}(z,w)=\frac{w^{1/2}}{z^{1/2}}\cdot\frac{((uv)^2;(uv)^2)_\infty^2}{(uz,-uw,-\frac{v}{z},\frac{v}{w};uv)_\infty}\cdot\frac{\theta_{(uv)^2}(u^2zw)}{\theta_{(uv)^2}(\frac{w}{z})}\cdot\frac{\theta_3\left((\frac{tz}{w})^2;(uv)^4\right)}{\theta_3(t^2;(uv)^4)}$$

$$\kappa_{2,2}(z,w)=\frac{tv^2}{z^{1/2}w^{3/2}}\cdot\frac{((uv)^2;(uv)^2)_\infty^2}{(-uz,-uw,\frac{v}{z},\frac{v}{w};uv)_\infty}\cdot\frac{\theta_{(uv)^2}(\frac{w}{z})}{\theta_{(uv)^2}(u^2zw)}\cdot\frac{\theta_3\left((\frac{tv^2}{zw})^2;(uv)^4\right)}{\theta_3(t^2;(uv)^4)}$$

- $(a_1,\dots,a_m;q)_\infty := \prod_{k=0}^\infty (1-a_1q^k)\cdots(1-a_mq^k)$ is the infinite q-Pochhammer symbol with multiple arguments
- $\theta_q(z) := (z;q)_\infty(q/z;q)_\infty$ is the "multiplicative" theta function
- $\theta_3(z;q) := \sum_{n\in\mathbb{Z}} q^{n^2/2}z^n$- the Jacobi theta function

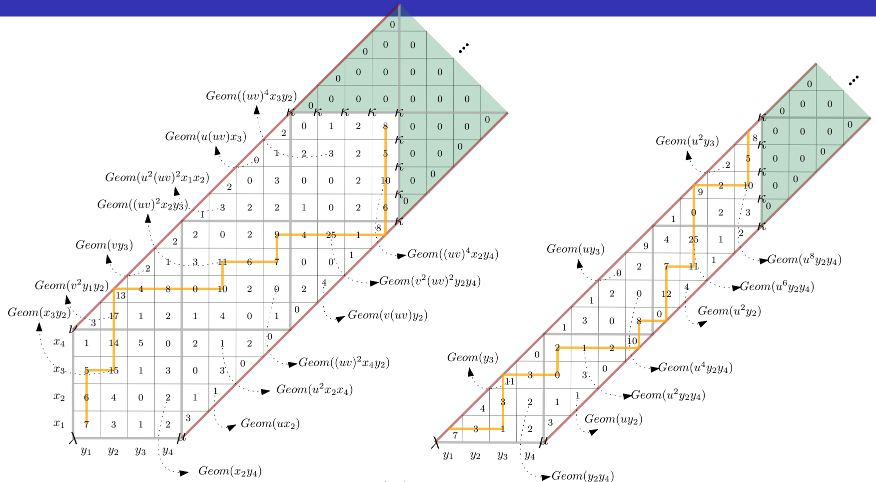
Outline

Random growth diagram/LPP model



- bijection: interlaced partitions $\to$ (admissible fillings, ν^{∞})
- $T(i,j)$ - the LPP time from (i,j) to ∞
- $\lambda_1(i,j) = T(i,j) + \nu_1^{\infty}$

Tie models



- full tie: $P(\kappa) = (uv; uv)_\infty (uv)^{|\kappa|}$
  $P^{\nearrow\searrow}(\mu, \lambda, \nu) = u^{|\mu|} v^{|\nu|} s_{\lambda/\nu}(x_1, \ldots, x_n) s_{\lambda/\mu}(y_1, \ldots, y_n)$

- half tie: $Prob(\kappa) = (u; u)_\infty (u)^{|\kappa|}$
  $Prob^{\nearrow}(\mu, \lambda) \propto u^{|\mu|} s_{\lambda/\mu}(y_1, \ldots y_n)$

Outline

Two-free boundary poissonized Plancherel model

- measures that interpolate between poissonized Plancherel and uniform measure on partitions ($\epsilon \to 0$):

$$\mathbb{M}^{\nearrow\searrow}(\mu, \lambda, \nu) \propto u^{|\mu|} v^{|\nu|} \cdot \frac{\epsilon^{|\lambda/\mu|+|\lambda/\nu|} f^{\lambda/\mu} f^{\lambda/\nu}}{|\lambda/\mu|! \cdot |\lambda/\nu|!}$$

$$\mathbb{M}^{\nearrow}(\mu, \lambda) \propto u^{|\mu|} \cdot \frac{\epsilon^{|\lambda/\mu|} f^{\lambda/\mu}}{|\lambda/\mu|!}$$

- $f^{\lambda/\mu}$ is the number of standard Young tableux of shape λ/μ.
- $\mathbb{M}^{\nearrow\searrow}$ Baik-Deift-Johansson [1999] when $u = v = 0$.
- $\mathbb{M}^{\nearrow}$ Baik-Rains [2000] when $u = 0$.
- λ_1 or LPP time has a (new) probability distribution that generalizes the classical Tracy-Widom GUE, GOE and GSE distributions.

Two-free boundary geometric model

- measures that interpolate between Johansson's corner growth measure and uniform measure on partitions ($q \to 0$):

$$\widetilde{\mathbb{M}}^{\nearrow\searrow}(\mu,\lambda,\nu) \propto u^{|\mu|}v^{|\nu|} \cdot q^{|\lambda/\mu|+|\lambda/\nu|}\tilde{f}^{n,\lambda/\mu}\tilde{f}^{n,\lambda/\nu}$$
$$\widetilde{\mathbb{M}}^{\nearrow}(\mu,\lambda) \propto u^{|\mu|} \cdot q^{|\lambda/\mu|}\tilde{f}^{n,\lambda/\mu}$$

- $\tilde{f}^{\lambda/\mu}$ is the number of semi-standard Young tableux of shape λ/μ.
- $\widetilde{\mathbb{M}}^{\nearrow\searrow}$ Johansson [2000] when $u = v = 0$.
- $\widetilde{\mathbb{M}}^{\nearrow}$ Baik-Rains [2000] when $u = 0$.

Theorem (Poissonized Plancherel)

$\eta > 0$ is fixed, $M := \frac{\epsilon}{1-u^2} \to \infty$, $u = v = \exp(-\eta M^{-1/3}) \to 1 (\epsilon \sim M^{2/3})$

$$\lim_{M\to\infty} \mathbb{M}^{\nearrow\searrow}\Big(\frac{\lambda_1 - 2M}{M^{1/3}} \le s + \frac{1}{2\eta}\log\frac{M^{1/3}}{2\eta}\Big) = F^{\nearrow\searrow;\eta}(s)$$

$$\lim_{M\to\infty} \mathbb{M}^{\nearrow}\Big(\frac{\lambda_1 - 2M}{M^{1/3}} \le s + \frac{1}{\eta}\log\frac{M^{1/3}}{\eta}\Big) = F^{\nearrow;\eta}(s)$$

Theorem (Geometric)

$\eta > 0$ is fixed, $n \to \infty$, $u = v = \exp(-\eta n^{-1/3}) \to 1$, $q = 1 - u^2 \to 0$

$$\lim_{n\to\infty} \widetilde{\mathbb{M}}^{\nearrow\searrow}\Big(\frac{\lambda_1 - \chi n}{n^{1/3}} \le s + \frac{1}{2\eta}\log\frac{n^{1/3}}{2\eta}\Big) = F^{\nearrow\searrow;\eta}(s)$$

$$\lim_{n\to\infty} \widetilde{\mathbb{M}}^{\nearrow}\Big(\frac{\lambda_1 - \chi n}{n^{1/3}} \le s + \frac{1}{\eta}\log\frac{n^{1/3}}{\eta}\Big) = F^{\nearrow;\eta}(s)$$

where $\chi = 2q\sum_{\ell\ge 0}\frac{u^{2\ell}}{1-u^{2\ell}q} \xrightarrow{n\to\infty} 2$.

$$\lim_{\eta\to\infty} F^{\nearrow\searrow;\eta}(s) = F_{GUE}(s)$$

$$F^{\nearrow\searrow;\eta}\left(s-\frac{\log 2}{2\eta}\right) = \mathrm{pf}\,(J-A^{\eta})_{L^2(s,\infty)}$$

$$A^{\eta}_{1,1}(x,y) = \int\int \Gamma\left(\frac{1}{2}-\frac{\zeta}{2\eta},\frac{1}{2}-\frac{\omega}{2\eta}\right)\frac{\sin\frac{\pi(\zeta-\omega)}{4\eta}}{\cos\frac{\pi(\zeta+\omega)}{4\eta}} e^{\frac{\zeta^3}{3}-x\zeta+\frac{\omega^3}{3}-y\omega}\frac{\mathrm{d}_{\zeta\omega}}{4\eta}$$

$$A^{\eta}_{1,2}(x,y) = \int\int \Gamma\left(\frac{1}{2}-\frac{\zeta}{2\eta},\frac{1}{2}+\frac{\omega}{2\eta}\right)\frac{\cos\frac{\pi(\zeta+\omega)}{4\eta}}{\sin\frac{\pi(\zeta-\omega)}{4\eta}} e^{\frac{\zeta^3}{3}-x\zeta-\frac{\omega^3}{3}+y\omega}\frac{\mathrm{d}_{\zeta\omega}}{4\eta}$$

$$A^{\eta}_{2,2}(x,y) = \int\int \Gamma\left(\frac{1}{2}+\frac{\zeta}{2\eta},\frac{1}{2}+\frac{\omega}{2\eta}\right)\frac{\sin\frac{\pi(\zeta-\omega)}{4\eta}}{\cos\frac{\pi(\zeta+\omega)}{4\eta}} e^{-\frac{\zeta^3}{3}+x\zeta-\frac{\omega^3}{3}+y\omega}\frac{\mathrm{d}_{\zeta\omega}}{4\eta}$$

where $A^{\eta}_{2,1}(x,y) := -A^{\eta}_{1,2}(y,x)$, $\Gamma(u,v) = \Gamma(u)\Gamma(v)$ and $\mathrm{d}_{\zeta\omega} = \mathrm{d}\zeta\mathrm{d}\omega/(2\pi i)^2$

Bulk limit

homogenous model:

- as a Schur process

$$\mathrm{P}rob(\vec{\lambda},\vec{\mu}) \propto \prod_{k=1}^{N} a^{\frac{|\lambda^{(k-1)}|+|\lambda^{(k)}|}{2}} (ba^{-1})^{|\mu^{(k)}|}$$
$$\propto \prod_{k=1}^{N} s_{\lambda^{(k)}/\mu^{(k-1)}}(a^{1/2}) b^{|\mu^{(k)}|} s_{\lambda^{(k)}/\mu^{(k)}}(a^{1/2})$$

- as the random growth diagram:

$$n(i,j) \sim \begin{cases} \mathrm{G}eom(ab^{i+j}) & (i,j) \notin \mathrm{r.l.} \\ \mathrm{G}eom(a^{1/2}b^{(i+j)/2}) & (i,j) \in \mathrm{r.l.} \end{cases} \qquad Prob(\nu^{\infty}) \propto b^{N|\nu^{\infty}|}$$

Plancherel limit: $a^{1/2} = Lr, b = e^{-r}, r \to 0$

Theorem

Conditions: $L = \frac{\gamma}{1-e^{-2r}}$, $rk, rk' \to \kappa$, $0 < \frac{t}{r} \to \tau < 1$ *and* $0 < \frac{t'}{r} \to \tau' < 1$, *where* γ, κ, $k - k'$, τ *and* τ' *are constant.*

$$\lim_{r\to 0} K_{1,2}(t,k;t',k') = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos((k'-k)\phi)\frac{e^{(\tau'-\tau)(\kappa-\gamma\cos\phi)}}{1+\tan^2\frac{\phi}{2}e^{2\kappa-2\gamma\cos\phi}}d\phi$$

when $\tau \leq \tau'$.

$$\lim_{r\to 0} K_{1,1}(t,k;t',k') = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin((k'-k)\phi)\cot\frac{\phi}{2}\frac{e^{(-\tau'-\tau)(\kappa-\gamma\cos\phi)}}{1+\tan^2\frac{\phi}{2}e^{2\kappa-2\gamma\cos\phi}}d\phi$$

$$\lim_{r\to 0} K_{2,2}(t,k;t',k') = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin((k'-k)\phi)\tan\frac{\phi}{2}\frac{e^{(\tau'+\tau)(\kappa-\gamma\cos\phi)}}{1+\tan^2\frac{\phi}{2}e^{2\kappa-2\gamma\cos\phi}}d\phi$$

- $\gamma = 0$: uniform measure with $K = \frac{1}{1+e^{2\kappa}}$ [Okounkov 1999]
- $\gamma \to \infty$ and κ proportional to γ: poissonized Plancherel measure [Borodin,Okonukov, Olshanski 1999] - discrete sine kernel

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Thank you.